Positivity, decay, and extinction for a singular diffusion equation with gradient absorption

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Abstract

We study qualitative properties of non-negative solutions to the Cauchy problem for the fast diffusion equation with gradient absorption

$$\partial_t u - \Delta_p u + |\nabla u|^q = 0 \quad \text{in} \quad (0,\infty)\times\mathbb{R}^N,$$

where $N \geq 1$, $p \in (1,2)$, and $q > 0$. Based on gradient estimates for the solutions, we classify the behavior of the solutions for large times, obtaining either positivity as $t \to \infty$ for $q > p - N/(N+1)$, optimal decay estimates as $t \to \infty$ for $p/2 \leq q \leq p - N/(N+1)$, or extinction in finite time for $0 < q < p/2$. In addition, we show how the diffusion prevents extinction in finite time in some ranges of exponents where extinction occurs for the non-diffusive Hamilton-Jacobi equation.

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1 Introduction

In this paper we study qualitative properties of the non-negative continuous solutions to the following equation with singular diffusion and gradient absorption

$$\partial_t u - \Delta_p u + |\nabla u|^q = 0, \quad (t,x) \in Q_\infty := (0,\infty) \times \mathbb{R}^N, \tag{1.1}$$

where we consider $1 < p < 2$, $q > 0$ and a non-negative initial condition

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^N. \tag{1.2}$$

As usual, the p-Laplacian operator is defined by

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

Equation (1.1), when $p \in (1,2)$, is a quasilinear singular diffusion equation (also known in the literature as the *fast p-Laplacian equation*), with a nonlinear absorption term depending on the euclidean norm of the gradient. In recent years, both the semilinear problem ($p = 2$) and the degenerate diffusion-absorption problem ($p > 2$) have been investigated, with emphasis on the large time behavior. It has been noticed that the asymptotic behavior as $t \to \infty$ depends strongly on the value of $q > 0$, and for $p = 2$ there are many results available, see for example [1, 4, 5, 6, 7, 9, 12, 13]. From all these results, an almost complete understanding of the large time behavior for the semilinear case $p = 2$ is now available. In particular, finite time extinction takes place for $q \in (0,1)$ while the dynamics is either solely dominated by the diffusion or is the result of a balance between the diffusion and the absorption according to the value of $q > 1$.

More recently, the research has been extended to the degenerate case $p > 2$. In this range, the situation is very different: indeed, on the one hand, the support of compactly supported solutions advances in time with finite speed and interfaces appear [2]. On the other hand, there is a range of values of the parameter q, namely $q \in (1, p-1]$, where the dynamics of (1.1)-(1.2) is solely governed by the gradient absorption [17, 23], a feature which cannot be observed in the semilinear case ($p = 2$) for $q > 1$.

The purpose of this paper is to investigate the range $p \in (1,2)$, called fast p-Laplacian diffusion, where the diffusion is no longer degenerate but becomes singular when ∇u vanishes. This case turns out to be more complicated and we first point out that, even in the case of the diffusion equation

$$\partial_t \Phi - \Delta_p \Phi = 0 \quad \text{in} \quad Q_\infty\,, \tag{1.3}$$

important advances have been performed very recently, both in constructing special solutions with optimal decay estimates, see [18, 25] and in understanding regularity, smoothing effects and other deep qualitative properties of the solutions [11]. All this previous knowledge is a good starting point to investigate the competition between the fast p-Laplacian diffusion and the gradient absorption terms. The behavior of non-negative solutions Φ to the diffusion equation (1.3) and of non-negative solutions h to the Hamilton-Jacobi equation

$$\partial_t h + |\nabla h|^q = 0 \quad \text{in} \quad Q_\infty \tag{1.4}$$

indeed differs markedly: in particular, starting from a compactly supported initial condition, Φ becomes instantaneously positive in Q_∞ if $p \geq 2N/(N+1)$ while the support of h

stays the same for all times if $q > 1$ or becomes empty after a finite time if $q \in (0, 1]$. It is thus of interest to figure out how these two mechanisms compete in (1.1).

More specifically, the aim of this paper is to give a complete picture of the qualitative properties of non-negative solutions to (1.1)-(1.2), with respect to the following three types of behaviors: either the solution remains positive in the limit, or it decays to zero as $t \to \infty$ but is positive for finite times, or finally it extinguishes after a finite time. In fact, we describe the ranges, with respect to p and q, where these phenomena occur, and we also provide, in the cases where this is possible, a quantitative measure of how the solution behaves, providing estimates of decay rates or extinction rates.

The main tool for establishing such qualitative properties turns out to be gradient estimates having generally the form

$$\|\nabla u^{\gamma}(t)\|_{\infty} \leq C\|u_0\|_{\infty}^{\delta} t^{-\beta}, \tag{1.5}$$

for suitable exponents γ, $\delta > 0$, and $\beta > 0$. Such gradient estimates have been obtained in [3, 14] for $p = 2$ and $q > 0$ and in [2] for $p > 2$ and $q > 1$ by a Bernstein technique adapted from [8], the exponent γ depending on p and q and ranging in $(0, 1)$ for $p \geq 2$ and $q > 1$. This last property is of great interest as such estimates are clearly stronger than an estimate on $\|\nabla u(t)\|_{\infty}$ and are at the basis of the subsequent studies of the qualitative behavior of solutions to (1.1) for $p \geq 2$. We shall establish similar gradient estimates for (1.1) when p and q range in $(1, 2)$ and $(0, \infty)$, respectively. A particularly interesting new feature is that the singular diffusion allows us to obtain gradient estimates with negative exponents γ. As we shall see below, these estimates have clearly a link with the positivity properties of the solutions to (1.1) which are expected when the diffusion dominates.

Notion of solution. Owing to the nonlinear reaction term $|\nabla u|^q$ involving the gradient of u, a suitable notion of solution for Equation (1.1) is that of *viscosity solution*. Due to the singular character of (1.1) at points where ∇u vanishes, the standard definition of viscosity solution has to be adapted to deal with this case [19, 20, 24]. In fact, it requires to restrict the class of comparison functions [19, 24] and we refer to Definition 6.1 for a precise definition. A remarkable feature of this modified definition is that basic results about viscosity solutions, such as comparison principle and stability property, are still valid, see [24, Theorem 3.9] (comparison principle) and [24, Theorem 6.1] (stability). The relationship between viscosity solutions and other notions of solutions is investigated in [20]. From now on, by a solution to (1.1)-(1.2) we mean a viscosity solution in the sense of Definition 6.1 below.

Main results.

For later use, we introduce the following notations for the critical exponents

$$p_c := \frac{2N}{N+1}, \ p_{sc} := \frac{2(N+1)}{N+3}, \ q_{\star} := p - \frac{N}{N+1} \tag{1.6}$$

and for several constants

$$\begin{gathered} k := \frac{(2-p)[p(N+3)-2(N+1)]}{4(p-1)}, \ \xi := \frac{1}{q(N+1)-N}, \ \eta := \frac{1}{N(p-2)+p}, \\ q_1 := \max\left\{p-1, \frac{N}{N+1}\right\}, \end{gathered} \tag{1.7}$$

appearing frequently in our analysis. Throughout the paper, C, C', and C_i, $i \geq 1$, denote constants depending only on N, p, and q. The dependence of these constants upon additional parameters will be indicated explicitly.

Let us begin with basic decay estimates which are valid for general non-negative Lipschitz continuous and integrable initial data without any extra conditions.

Theorem 1.1. *Assume that*

$$u_0 \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N), \quad u_0 \geq 0, \ u_0 \not\equiv 0. \tag{1.8}$$

Then there exists a unique non-negative (viscosity) solution u to (1.1)-(1.2) *such that:*

(i) if $p > p_c$ and $q > q_\star$, then

$$\|u(t)\|_\infty \leq C \ \|u_0\|_1^{p\eta} \ t^{-N\eta}, \quad t > 0. \tag{1.9}$$

(ii) if $p > p_c$ and $q \in (N/(N+1), q_\star]$, then

$$\|u(t)\|_\infty \leq C \ \|u_0\|_1^{q\xi} \ t^{-N\xi}, \quad t > 0. \tag{1.10}$$

(iii) if $p > p_c$ and $q = N/(N+1)$ or $p = p_c$ and $q \geq p_c/2$, then

$$\|u(t)\|_\infty \leq C'(u_0) \ e^{-C(u_0)t}, \quad t > 0. \tag{1.11}$$

(iv) if $p \geq p_c$ and $q \in (0, N/(N+1))$ or $p \in (1, p_c)$, then there is $T_e > 0$ depending only on N, p, q, and u_0 such that

$$u(t,x) \equiv 0, \quad (t,x) \in [T_e, \infty) \times \mathbb{R}^N. \tag{1.12}$$

Let us first mention that the main contribution of Theorem 1.1 is not the existence and uniqueness of a viscosity solution to (1.1)-(1.2), as the latter readily follows from the comparison principle [24, Theorem 3.9] while the former is likely to be proved by Perron's method such as in [24, Section 4]. We shall however provide a proof in the final section as it is needed in order to justify the derivation of the gradient estimates stated below. Next, we notice that the decay estimates (1.9) and (1.10) are also enjoyed by non-negative and integrable solutions to (1.3) and (1.4), respectively. Since $t^{-N\xi} \leq t^{-N\eta}$ for $t \geq 1$ and $q < q_\star$, Theorem 1.1 already uncovers a dichotomy in the behavior of solutions to (1.1)-(1.2) for $p \geq p_c$ with a faster decay induced by the absorption term for $q < q_\star$. This decay is even faster for $q \in (0, N/(N+1)]$. Still, as we shall see now, more precise information can be obtained for initial data with a fast decay at infinity and the first main result of this paper is the following improvement of Theorem 1.1 for $p > p_c$.

Theorem 1.2. *Assume that u_0 satisfies* (1.8)*. Then the corresponding solution u to* (1.1)-(1.2) *satisfies:*

(i) if $p \in (p_c, 2)$, $q \in (p/2, q_\star)$, and there is $C_0 > 0$ such that

$$u_0(x) \leq C_0 \ |x|^{-(p-q)/(q-p+1)}, \quad x \in \mathbb{R}^N, \tag{1.13}$$

then

$$t^{(N+1)(q_\star-q)/(2q-p)} \ \|u(t)\|_1 + t^{(p-q)/(2q-p)} \ \|u(t)\|_\infty \leq C(u_0), \quad t > 0. \tag{1.14}$$

(ii) if $p \in (p_c, 2)$, $q = p/2$, and u_0 satisfies (1.13), *then*

$$\|u(t)\|_1 + \|u(t)\|_\infty \le C'(u_0)\ e^{-C(u_0)t}\,, \quad t > 0\,. \tag{1.15}$$

(iii) if $p \in (p_c, 2)$, $q \in (0, p/2)$, and there are $C_0 > 0$ and $Q > 0$ such that

$$u_0(x) \le C_0\ |x|^{-(p-Q)/(Q-p+1)}\,, \quad x \in \mathbb{R}^N\,, \tag{1.16}$$

with $Q = q$ if $q \in (q_1, p/2)$ and $Q \in (q_1, p/2)$ if $q \in (0, q_1]$. Then there is $T_e > 0$ depending only on N, p, q, and u_0 such that

$$u(t,x) \equiv 0\,, \quad (t,x) \in [T_e, \infty) \times \mathbb{R}^N\,. \tag{1.17}$$

Noting that $(p-q)/(2q-p) > N\xi$ for $q \in (0, q_\star)$, the decay estimates obtained in Theorem 1.2 are clearly faster than those of Theorem 1.1 for initial data decaying sufficiently rapidly as $|x| \to \infty$.

Let us next notice that a very interesting point in the previous theorem is the appearance of a new critical exponent for the absorption, $q = p/2$, that in the slow-diffusion range $p > 2$ did not play any role. Moreover, this critical exponent is a branching point for the behavior, as an interface between decay as $t \to \infty$ and finite time extinction. It is worth mentioning that the corresponding critical exponent for $p > 2$ is $q = p - 1$ and that we have $p - 1 = p/2 = 1$ exactly when $p = 2$.

Another interesting remark related to Theorem 1.2 is the fact that, for $p \in [p_c, 2)$ and $q \in [p/2, 1)$, the diffusion prevents extinction in finite time, see Proposition 1.8 below. This is a feature which matches with the linear diffusion case $p = 2$, since, under suitable conditions on the initial data u_0, finite time extinction could appear for any $q \in (0, 1)$ [5, 6, 13].

As mentioned above, the key technical tool for studying the large time behavior of the solutions of (1.1) is the availability of suitable gradient estimates, with abstract form (1.5). Their proof relies on a Bernstein technique borrowing ideas from [8] and, apart from their technical interest in the proof of our main theorem, they are interesting by themselves. Let us first denote the positivity set $\mathcal{P}$ of u by

$$\mathcal{P} := \{(t,x) \in Q_\infty :\ u(t,x) > 0\}. \tag{1.18}$$

Theorem 1.3. *Let $p > p_c$ and u_0 satisfy* (1.8). *The corresponding solution u to* (1.1)-(1.2) *satisfies the following gradient estimates:*

(i) for $q \in [1, \infty)$, we have

$$\left|\nabla u^{-(2-p)/p}(t,x)\right| \le \left(\frac{2-p}{p}\right)^{(p-1)/p} \eta^{1/p} t^{-1/p},\ (t,x) \in \mathcal{P}. \tag{1.19}$$

(ii) for $q \in [p/2, 1)$, we have

$$|\nabla u^{-(2-p)/p}(t,x)| \le C\left(\|u_0\|_\infty^{(2q-p)/p(p-q)} + t^{-1/p}\right),\ (t,x) \in \mathcal{P}. \tag{1.20}$$

(iii) for $q \in (p-1, p/2)$, we have

$$|\nabla u^{-(q-p+1)/(p-q)}(t,x)| \le C\left(1 + \|u_0\|_\infty^{(p-2q)/p(p-q)} t^{-1/p}\right),\ (t,x) \in \mathcal{P}. \tag{1.21}$$

(iv) for $q = p-1$, we have the logarithmic estimate

$$|\nabla \log u(t,x)| \le C \left(1 + \|u_0\|_\infty^{(2-p)/p} t^{-1/p}\right), \ (t,x) \in \mathcal{P}. \tag{1.22}$$

(v) for $q \in (0, p-1)$, we have

$$|\nabla u^{(p-q-1)/(p-q)}(t,x)| \le C \left(1 + \|u_0\|_\infty^{(p-2q)/p(p-q)} t^{-1/p}\right), \ (t,x) \in Q_\infty. \tag{1.23}$$

A striking feature in Theorem 1.3 is that in parts (i)-(iii) gradients of negative powers of the solutions appear. Besides being seemingly new, these estimates are rather unusual and obviously stronger than an estimate for only $|\nabla u|$, which can be easily deduced from them. They are valid only on the positivity set of u but, as we shall show below, $\mathcal{P}$ coincides with Q_∞ when $p \ge p_c$ and $q \ge p/2$, and $\mathcal{P} \subseteq (0, T_e) \times \mathbb{R}^N$ for $1 < p < p_c$ or $p_c \le p < 2$ and $q < p/2$, for some $T_e < \infty$.

Remark 1.4. *We actually prove a stronger result, namely that, for any $\delta > 0$, $|\nabla(u+\delta)^{-(2-p)/p}(t,x)|$ (respectively $|\nabla(u+\delta)^{-(2-p)/p}(t,x)|$, $|\nabla(u+\delta)^{-(q-p+1)/(p-q)}(t,x)|$ and $|\nabla \log(u+\delta)(t,x)|$) is bounded by the same right-hand side as in* (1.19) *(respectively* (1.20), (1.21) *and* (1.22)*) for all $(t,x) \in Q_\infty$. For instance, for $q \in [1,\infty)$ we have*

$$\left|\nabla(u+\delta)^{-(2-p)/p}(t,x)\right| \le \left(\frac{2-p}{p}\right)^{(p-1)/p} \eta^{1/p} t^{-1/p}, \ (t,x) \in Q_\infty. \tag{1.24}$$

As the right-hand side of (1.24) *does not depend on $\delta > 0$, we deduce* (1.19) *by letting $\delta \to 0$ wherever it is possible, that is in $\mathcal{P}$.*

These gradient estimates will be used in the sequel to prove parts of Theorem 1.2. Their proof is divided into two parts and performed in Sections 2.1 and 2.2.

We obtain similar gradient estimates for $p = p_c$ and $p < p_c$. In the case $p = p_c$ being a critical exponent, some logarithmic corrections appear in the gradient estimates; they are gathered in the following result, that is proved in Section 2.3. Notice that, as $p_c = 1$ in one space dimension, the next theorem is only valid for $N \ge 2$.

Theorem 1.5. *Let $p = p_c$ and u_0 satisfy* (1.8)*. The corresponding solution u to* (1.1)-(1.2) *satisfies the following gradient estimates:*

(i) for $q \ge 1$ and $(t,x) \in \mathcal{P}$, we have

$$|\nabla u^{-1/N}(t,x)| \le C \left(\log\left(\frac{e\|u_0\|_\infty}{u(t,x)}\right)\right)^{1/p_c} t^{-1/p_c}. \tag{1.25}$$

(ii) for $q \in (N/(N+1), 1)$ and $(t,x) \in \mathcal{P}$, we have

$$|\nabla u^{-1/N}(t,x)| \le C \left(\|u_0\|_\infty^{1/N\xi(p_c-q)} + t^{-1/p_c}\right) \left(\log\left(\frac{e\|u_0\|_\infty}{u(t,x)}\right)\right)^{1/p_c}. \tag{1.26}$$

(iii) for $q = N/(N+1) = p_c/2$ and $(t,x) \in \mathcal{P}$, we have

$$|\nabla u^{-1/N}(t,x)| \le C \left(\log\left(\frac{e\|u_0\|_\infty}{u(t,x)}\right)\right)^{2/p_c} \left(1 + t^{-1/p_c}\right). \tag{1.27}$$

(iv) for $q \in (0, N/(N+1))$, the previous gradient estimates (1.21)*,* (1.22) *and* (1.23) *still hold true.*

Remark 1.6. *Similarly to the case $p > p_c$ (recall Remark 1.4), given $\delta > 0$, the estimates* (1.25)-(1.27) *are true for all $(t, x) \in Q_\infty$ provided that $u(t, x)$ is replaced by $u(t, x) + \delta$ on both sides of the inequalities.*

In the range $p < p_c$, the situation becomes more technical and more involved, and apparently there is a new critical exponent coming from the diffusion that plays a role, $p_{sc} = 2(N+1)/(N+3)$. We can still establish gradient estimates for this range, but it requires to handle separately several cases according to the value of q. Since they are not used afterwards, we do not state nor prove them but refer the interested reader to Section 2.4 where we provide a proof only for a limited range of q, namely, $q \geq 1 - k$.

Finally, another useful gradient estimate is the one which retains only the influence of the Hamilton-Jacobi term:

Theorem 1.7. *Let $p \in [p_c, 2)$ and u_0 satisfy* (1.8)*. The corresponding solution u to* (1.1)-(1.2) *satisfies the following gradient estimates: if $q \in (0, 1)$, we have*

$$|\nabla u(t, x)| \leq C \|u_0\|_\infty^{1/q} t^{-1/q}, \ \ (t, x) \in Q_\infty, \tag{1.28}$$

while, if $q > 1$, we have a slightly better formulation:

$$\left|\nabla u^{(q-1)/q}(t, x)\right| \leq \frac{1}{q}(q-1)^{(q-1)/q} t^{-1/q}, \ \ (t, x) \in Q_\infty. \tag{1.29}$$

These estimates are proved by similar modified Bernstein techniques, but their main difference with respect to the previous ones is that it is the term coming from the diffusion which is simply discarded. They actually hold in more general ranges of p as we can deduce by analyzing their proof in Section 2.6.

Having discussed the occurrence of finite time extinction in Theorems 1.1 and 1.2 and obtained gradient estimates valid on the positivity set (1.18) of u, we finally turn to the positivity issue: we first observe that the L^1-norm of solutions u to (1.1)-(1.2) is non-increasing. It thus has a limit as $t \to \infty$ which is non-negative and it is natural to wonder whether the absorption term may drive it to zero as $t \to \infty$ or not. This question is obviously only meaningful for $p \geq p_c$ for which there is no extinction for the diffusion equation (1.3) but conservation of mass [16]. In this direction, we also prove the following positivity result that completes the panorama given in Theorem 1.2.

Proposition 1.8. *Let $p \in [p_c, 2)$, u_0 satisfy* (1.8)*, and u be the solution to* (1.1)-(1.2)*.*

(1) If either $p > p_c$ and $q \geq p/2$ or $p = p_c$ and $q > p_c/2$, then $\|u(t)\|_1 > 0$ for all $t \geq 0$ and the positivity set satisfies $\mathcal{P} = Q_\infty$.

(2) We have $\lim\limits_{t \to \infty} \|u(t)\|_1 > 0$ if and only if $q > q_\star$.

Thanks to Theorem 1.2 and Proposition 1.8, we thus have a clear separation between positivity and finite time extinction, the latter occurring when either $p \geq p_c$ and $q \in (0, p/2)$ or $p \in (1, p_c)$ while the former is true in Q_∞ for $p \geq p_c$ and $q \geq p/2$. Let us emphasize that, for $p \in [p_c, 2)$ and $q \in [p/2, 1)$, the diffusion term prevents the finite time

	$0 < q < p/2$	$q = p/2$	$p/2 < q < q_\star$	$q_\star \le q$
$p \in [p_c, 2)$	extinction	positivity exponential decay	positivity fast algebraic decay	positivity diffusion decay
$p \in (1, p_c)$	extinction	extinction	extinction	extinction

Table 1: Behavior of u for initial data decaying sufficiently fast at infinity

extinction that would occur in the absence of diffusion. Table 1 provides a summary of the outcome of this paper.

Organization of the paper. A formal proof of the gradient estimates for solutions to (1.1) is given in Section 2, which is divided into several subsections according to the range of the exponents p and q. Then, a rigorous approach by approximation and regularization, completing the formal one and settling also the existence and uniqueness of solutions to (1.1)-(1.2) is appended, due to its highly technical character, see Section 6. We prove Theorem 1.1 in Section 3. Before proving our main Theorem 1.2, we devote Section 4 to the behavior of the L^1-norm of u as $t \to \infty$ and to the positivity issue as well. Finally, we prove our main Theorem 1.2, together with Proposition 1.8, in Section 5.

2 Gradient estimates

As already mentioned, the proof of the gradient estimates relies on a Bernstein technique [8], also used in [2, 3, 14] for $p \ge 2$, but in the case $p \in (1,2)$ the technical details are quite different. We first have the following technical general lemma.

Lemma 2.1. *Let $p \in (1,2)$, $q > 0$, and consider a C^3-smooth monotone function φ. Set $v := \varphi^{-1}(u)$ and $w := |\nabla v|^2$, where u is a solution of (1.1). Then, the function w satisfies the following differential inequality:*

$$\partial_t w - Aw - B \cdot \nabla w + R \le 0 \quad \text{in} \quad Q_\infty, \tag{2.1}$$

where B is given in [2, Appendix A, Eq. (A.2)], and

$$Aw := |\nabla u|^{p-2} \Delta w + (p-2)|\nabla u|^{p-4} (\nabla u)^t D^2 w \nabla u, \tag{2.2}$$

$$R := 2(p-1) R_1 w^{(2+p)/2} + 2(q-1) R_2 w^{(2+q)/2}, \tag{2.3}$$

where R_1 and R_2 are given by

$$R_1 := |\varphi'|^{p-2} \left(k \left(\frac{\varphi''}{\varphi'} \right)^2 - \left(\frac{\varphi''}{\varphi'} \right)' \right) \tag{2.4}$$

(recall that k is defined in (1.7)) and

$$R_2 := |\varphi'|^{q-2} \varphi''. \tag{2.5}$$

We do not recall the precise form of B, since it is complicated and not needed in the sequel.

Proof. We begin with Lemma 2.1 in [2], which, by examining carefully the proof, holds true for monotone functions φ (not only for increasing functions, as stated in [2]). We obtain the differential inequality

$$\partial_t w - Aw - B\cdot\nabla w + 2\tilde{R}_1 w^2 + 2\tilde{R}_2 w \le 0,$$

where A and B have the form given in (2.2) and in [2, Eq. (A.2)], respectively, and

$$\tilde{R}_1 := -a\left(\frac{\varphi''}{\varphi'}\right)' - \left((N-1)\frac{(a')^2}{a} + 4a''\right)(\varphi'\varphi'')^2 w^2 - 2a'w(2(\varphi'')^2 + \varphi'\varphi'''),$$

$$\tilde{R}_2 := \frac{\varphi''}{(\varphi')^2}\left(2b'(\varphi')^2 w - b\right),$$

the dependence of a, a', a'', b, b' on $\varphi'(v)^2 w$ and of φ and its derivatives on v being omitted. In our case $a(r) = r^{(p-2)/2}$, $b(r) = r^{q/2}$. Using these formulas for a and b and the identity

$$\varphi'\varphi''' = \left(\frac{\varphi''}{\varphi'}\right)'(\varphi')^2 + (\varphi'')^2,$$

we compute $\tilde{R}_1$ and $\tilde{R}_2$ and obtain

$$\begin{aligned}
-\tilde{R}_1 &= (p-1)|\varphi'|^{p-2}\left(\frac{\varphi''}{\varphi'}\right)' w^{(p-2)/2} \\
&\quad + (p-2)\left[p-1+\frac{(N-1)(p-2)}{4}\right](\varphi'')^2\,|\varphi'|^{p-4}\,w^{(p-2)/2} \\
&= (p-1)|\varphi'|^{p-2}w^{(p-2)/2}\left[\left(\frac{\varphi''}{\varphi'}\right)' - k\left(\frac{\varphi''}{\varphi'}\right)^2\right] = -(p-1)w^{(p-2)/2}R_1.
\end{aligned}$$

and

$$\tilde{R}_2 = \frac{\varphi''}{(\varphi')^2}\left(q\,|\varphi'|^q\,w^{(q-2)/2}w - |\varphi'|^q\,w^{q/2}\right) = (q-1)R_2 w^{q/2}.$$

arriving to the formula (2.5). Let us notice that this is still a formal proof, since [2, Lemma 2.1] requires a and b to be C^2-smooth, and our choices are not. For a rigorous proof, we have to approximate a and b by their regularizations

$$a_\varepsilon(r) := (r+\varepsilon^2)^{(p-2)/2}, \quad b_\varepsilon(r) := (\varepsilon^2 + r)^{q/2} - \varepsilon^q,\ \varepsilon > 0,$$

and pass to the limit as $\varepsilon \to 0$, see Section 6. $\square$

We also introduce the function $\varrho := 1/\psi'$, where $\psi := \varphi^{-1}$. We have

$$\varphi'(v) = \varrho(u), \quad \varphi''(v) = (\varrho\varrho')(u),$$

hence, by straightforward calculations, we obtain the following alternative formulas for R_1 and R_2:

$$R_1 = |\varrho(u)|^{p-2}\left(k(\varrho'(u))^2 - (\varrho\varrho'')(u)\right) \tag{2.6}$$

and

$$R_2 = |\varrho(u)|^{q-2}\varrho(u)\varrho'(u). \tag{2.7}$$

We now choose in an appropriate way ϱ in equations (2.6) and (2.7), in order to have either $R_1 = 1$, $R_2 = 1$ or $R_1 = R_2$. In this way we obtain gradient estimates in the form of estimates for the function w in the notations of Lemma 2.1.

Let us notice at that point that, if we take $\varrho(z) \equiv 1$, we have $R_1 = R_2 = 0$ and $\varphi = \psi = \mathrm{Id}$; thus, $w = |\nabla u|^2$ satisfies the differential inequality

$$Lw := \partial_t w - Aw - B \cdot \nabla w \leq 0 \text{ in } Q_\infty.$$

Since $w(0) \leq \|\nabla u_0\|_\infty^2$ and the constant function $\|\nabla u_0\|_\infty^2$ is a solution for the operator L, by comparison we obtain

$$\|\nabla u(t)\|_\infty \leq \|\nabla u_0\|_\infty\,, \qquad t \geq 0\,. \tag{2.8}$$

2.1 Gradient estimates for $p > p_c$ and $q \geq p/2$

For this range of parameters, we choose

$$\varrho(z) = \left(\frac{p^2}{2(2k+p-2)}\right)^{1/p} z^{2/p}, \tag{2.9}$$

after noticing that

$$2k + p - 2 = \frac{(2-p)(N+1)(p-p_c)}{2(p-1)} > 0. \tag{2.10}$$

Then it is immediate to check that $R_1 = 1$ (in fact this is the way we discover this choice of ϱ) and

$$R_2 = \frac{2}{p}\left(\frac{p^2}{2(2k+p-2)}\right)^{q/p} u^{(2q-p)/p} \geq 0,$$

hence

$$R = 2(p-1)w^{(p+2)/2} + \frac{4(q-1)}{p}\left(\frac{p^2}{2(2k+p-2)}\right)^{q/p} u^{(2q-p)/p} w^{(q+2)/2}.$$

Case 1. For $q \geq 1$, $(q-1)R_2 \geq 0$, so that $R \geq 2(p-1)w^{(p+2)/2}$ and therefore

$$Lw := \partial_t w - Aw - B \cdot \nabla w + 2(p-1)w^{(p+2)/2} \leq 0. \tag{2.11}$$

Once established the differential inequality (2.11), the next step (that will be also used in the other cases) is to find a supersolution to the differential inequality (2.11) depending only on time, in this way avoiding the terms with the complicated forms of A and B. In our case, we notice that $W(t) := (p(p-1)t)^{-2/p}$ is a supersolution and conclude that

$$|\nabla v(t,x)| \leq (p(p-1)t)^{-1/p}, \ (t,x) \in Q_\infty.$$

But $v = \psi(u)$, hence $\nabla v = \psi'(u)\nabla u = \nabla u/\varrho(u)$; thus, substituting the value of ϱ, we obtain the inequality

$$|\nabla u(t,x)|\, u(t,x)^{-2/p} \leq \left[\frac{p^2}{2(2k+p-2)p(p-1)t}\right]^{1/p},$$

or equivalently (1.19).

Case 2. For $q \in [p/2, 1)$, the term coming from R_2 becomes negative and cannot be omitted. Instead, we will get the gradient estimate by compensating its negative effect with the positive term coming from R_1. Since $u(t,x) \le \|u_0\|_\infty$ for any $(t,x) \in Q_\infty$ and $2q - p > 0$, we have

$$\begin{aligned} R &= 2(p-1)w^{(p+2)/2} - \frac{4(1-q)}{p}\left(\frac{p^2}{2(2k+p-2)}\right)^{q/p} u^{(2q-p)/p} w^{(q+2)/2} \\ &\ge 2(p-1)w^{(q+2)/2}\left[w^{(p-q)/2} - \frac{4(1-q)}{p}\left(\frac{p^2}{2(2k+p-2)}\right)^{q/p} \|u_0\|_\infty^{(2q-p)/p}\right], \end{aligned}$$

hence

$$Lw := \partial_t w - Aw - B \cdot \nabla w + 2(p-1)w^{(q+2)/2}\left(w^{(p-q)/2} - c_1\right) \le 0, \tag{2.12}$$

where

$$c_1 := \frac{4(1-q)}{p}\left(\frac{p^2}{2(2k+p-2)}\right)^{q/p} \|u_0\|_\infty^{(2q-p)/p} > 0.$$

In a similar way as in the case $q \ge 1$, we notice that the function $W(t) := (2c_1)^{2/(p-q)} + (p(p-1)t/2)^{-2/p}$ is a supersolution for the partial differential operator L, hence

$$|\nabla v(t,x)| \le (2c_1)^{1/(p-q)} + \left(\frac{2}{p(p-1)t}\right)^{1/p}, \ (t,x) \in Q_\infty.$$

Since $|\nabla v| = |\nabla u|/\varrho(u)$, we deduce that there exists a constant $C > 0$ such that

$$\left|\nabla u^{-(2-p)/p}(t,x)\right| \le C\left(\|u_0\|_\infty^{(2q-p)/p(p-q)} + t^{-1/p}\right), \ (t,x) \in Q_\infty,$$

as stated in (1.20).

2.2 Gradient estimates for $p > p_c$ and $q < p/2$

In this case, we choose

$$\varrho(z) = \left(\frac{p-q}{k+p-q-1}\right)^{1/(p-q)} z^{1/(p-q)}, \tag{2.13}$$

noticing that

$$k + p - q - 1 = \frac{p}{2} - q + \frac{2k+p-2}{2} = \frac{p}{2} - q + \frac{(2-p)(N+1)(p-p_c)}{4(p-1)} > 0.$$

By straightforward calculations, it is immediate to check that

$$R_1 = R_2 = \varrho(u)^{q-1}\varrho'(u) = \frac{1}{p-q}\left(\frac{p-q}{k+p-q-1}\right)^{q/(p-q)} u^{(2q-p)/(p-q)} \ge 0,$$

so that

$$R = 2(p-1)R_2 w^{(q+2)/2}\left(w^{(p-q)/2} - \frac{1-q}{p-1}\right).$$

It follows that

$$Lw := \partial_t w - Aw - B\cdot\nabla w + 2(p-1)R_2 w^{(q+2)/2}\left(w^{(p-q)/2} - \frac{1-q}{p-1}\right) \le 0. \tag{2.14}$$

We next look for a supersolution of the form $W(t) = (2(1-q)/(p-1))^{2/(p-q)} + Kt^{-2/p}$, with K to be chosen depending on p, q, N, and $\|u_0\|_\infty$. Taking into account that $u(t,x) \le \|u_0\|_\infty$ for any $(t,x)\in Q_\infty$ and $2q-p<0$, we have

$$R_2 \ge \frac{1}{p-q}\left(\frac{p-q}{k+p-q-1}\right)^{q/(p-q)} \|u_0\|_\infty^{(2q-p)/(p-q)}$$

and

$$\begin{aligned}
LW &= -\frac{2}{p}Kt^{-1-(2/p)} + (p-1)R_2\left[W^{(p+2)/2} + W^{(q+2)/2}\left(W^{(p-q)/2} - 2\frac{1-q}{p-1}\right)\right] \\
&\ge -\frac{2}{p}Kt^{-1-(2/p)} + (p-1)R_2K^{(p+2)/2}t^{-1-(2/p)} \\
&\ge \frac{2K}{p}\left[\frac{p(p-1)}{2(p-q)}\left(\frac{p-q}{k+p-q-1}\right)^{q/(p-q)} \|u_0\|_\infty^{(2q-p)/(p-q)}K^{p/2} - 1\right] t^{-1-(2/p)}
\end{aligned}$$

hence, we find that $LW \ge 0$ provided that $K = C\|u_0\|_\infty^{2(p-2q)/p(p-q)}$ for some sufficiently large constant C. With this choice of K, the function W becomes a supersolution for L, and the comparison principle gives

$$|\nabla v(t,x)| \le C\left(1 + \|u_0\|_\infty^{(p-2q)/p(p-q)}t^{-1/p}\right),\ (t,x)\in Q_\infty,$$

or equivalently

$$|\nabla u(t,x)|u(t,x)^{-1/(p-q)} \le C\left(1 + \|u_0\|_\infty^{(p-2q)/p(p-q)}t^{-1/p}\right),\ (t,x)\in Q_\infty. \tag{2.15}$$

Thus, we have a discussion with respect to the sign of $p-1-q$. Indeed, if $q\in(p-1,p/2)$, we have

$$|\nabla u^{-(q-p+1)/(p-q)}(t,x)| \le C\left(1 + \|u_0\|_\infty^{(p-2q)/p(p-q)}t^{-1/p}\right),\ (t,x)\in Q_\infty.$$

If $q=p-1$, we have the logarithmic estimate

$$|\nabla \log u(t,x)| \le C\left(1 + \|u_0\|_\infty^{(p-2q)/p(p-q)}t^{-1/p}\right),\ (t,x)\in Q_\infty,$$

and if $q\in(0,p-1)$ we obtain a positive power estimate

$$|\nabla u^{(p-q-1)/(p-q)}(t,x)| \le C\left(1 + \|u_0\|_\infty^{(p-2q)/p(p-q)}t^{-1/p}\right),\ (t,x)\in Q_\infty.$$

This completes the proof of Theorem 1.3.

2.3 Gradient estimates for $p = p_c$ (and $N \geq 2$)

Case 1. Let us consider first $q > p_c/2 = N/(N+1)$. In this case, the constant k defined in (1.7) is given by $k = (2-p)/2 = 1/(N+1)$. By analogy with some gradient estimates obtained by Hamilton in [15] for the heat equation, we choose the following function:

$$\varrho(u) = u^{(N+1)/N}(\log M - \log u)^{(N+1)/2N}, \ M = e\|u_0\|_\infty.$$

Let us notice first that $\log M - \log u \geq 1$. Then, we obtain

$$\varrho'(u) = \frac{N+1}{2N}u^{1/N}\left[2\left(\log\frac{M}{u}\right)^{(N+1)/2N} - \left(\log\frac{M}{u}\right)^{-(N-1)/2N}\right]$$

and

$$\varrho''(u) = u^{-(N-1)/N}\left[\frac{N+1}{N^2}\left(\log\frac{M}{u}\right)^{(N+1)/2N} - \frac{(N+1)(N+2)}{2N^2}\left(\log\frac{M}{u}\right)^{-(N-1)/2N} \right.$$
$$\left. - \frac{(N+1)(N-1)}{4N^2}\left(\log\frac{M}{u}\right)^{-((N-1)/2N)-1}\right].$$

Hence, after an easy calculation, we have

$$k(\varrho'(u))^2 - \varrho(u)\varrho''(u) = u^{2/N}\left[\frac{N+1}{2N}\left(\log\frac{M}{u}\right)^{1/N} + \frac{N+1}{4N}\left(\log\frac{M}{u}\right)^{-(N-1)/N}\right],$$

which implies that

$$R_1 = \frac{N+1}{2N} + \frac{N+1}{4N}(\log M - \log u)^{-1} \geq \frac{N+1}{2N}.$$

On the other hand, calculating R_2, we find:

$$R_2 = \frac{N+1}{2N}u^{(q(N+1)-N)/N}\left[2\left(\log\frac{M}{u}\right)^{(N+1)q/2N} - \left(\log\frac{M}{u}\right)^{((N+1)q-2N)/2N}\right] > 0,$$

since $(\log M - \log u)^{-1} \leq 1 < 2$. Following the same division into cases with respect to q, we assume first that $q \geq 1$. In this case, we can simply omit the term coming from R_2, since $(q-1)R_2 \geq 0$, and end up with

$$R \geq \frac{N-1}{N}w^{(p_c+2)/2}.$$

Therefore

$$Lw := \partial_t w - Aw - B\cdot\nabla w + \frac{N-1}{N}w^{(p_c+2)/2} \leq 0. \tag{2.16}$$

Noticing that $W(t) = [(N+1)/(N-1)t]^{2/p_c}$ is a supersolution for L, we obtain that

$$|\nabla v(t,x)| \leq \left(\frac{N+1}{(N-1)t}\right)^{1/p_c}.$$

Coming back to the function u, this means

$$|\nabla u^{-1/N}(t,x)| \leq \frac{1}{N}\left(\frac{N+1}{N-1}\right)^{(N+1)/2N}(\log M - \log u(t,x))^{(N+1)/2N}t^{-(N+1)/2N}. \tag{2.17}$$

Case 2. Consider next $q \in (p_c/2, 1)$, In this case, we have to use again the strategy of compensation as in Section 2.1. First of all, we need to estimate R_2 from above. To this end, since $1/N\xi = [q(N+1) - N]/N < q(N+1)/2N$, we note that the function

$$z \mapsto z^{(q(N+1)-N)/N} (\log M - \log z)^{q(N+1)/2N}$$

attains its maximum over $(0, \|u_0\|_\infty)$ at $\|u_0\|_\infty e^{-(N\xi-1)/2} < \|u_0\|_\infty$. We deduce that

$$R_2 \leq \frac{N+1}{N} u^{(q(N+1)-N)/N} (\log M - \log u)^{(N+1)q/2N} \leq C_1 \|u_0\|_\infty^{(q(N+1)-N)/N},$$

hence

$$\begin{aligned} R &\geq \frac{N-1}{N} w^{(p_c+2)/2} - 2(1-q) C_1 \|u_0\|_\infty^{(q(N+1)-N)/N} w^{(q+2)/2} \\ &= \frac{N-1}{N} w^{(q+2)/2} \left(w^{(p_c-q)/2} - c_2 \right), \end{aligned}$$

where

$$c_2 = \frac{2N(1-q)C_1}{N-1} \|u_0\|_\infty^{(q(N+1)-N)/N}.$$

We now proceed as in Section 2.1 and notice that $W(t) = (2c_2)^{2/(p_c-q)} + [2(N+1)/(N-1)t]^{2/p_c}$ is a supersolution. By the comparison principle we obtain

$$|\nabla v(t,x)| \leq (2c_2)^{1/(p_c-q)} + \left(\frac{2(N+1)}{(N-1)t} \right)^{1/p_c}.$$

Going back to the definition of u, we find that

$$\frac{|\nabla u(t,x)|}{\varrho(u(t,x))} \leq C \left(\|u_0\|_\infty^{(q(N+1)-N)/N(p_c-q)} + t^{-(N+1)/2N} \right),$$

from which we deduce easily (1.26), taking into account the definition of ϱ.

Let us remark that this is an extension of the estimates that we obtain for $p > p_c$ and $q > p/2$, since for $p = p_c$, we have $(2-p)/p = 1/N$. Thus the negative power of the gradient is the same and the powers of t and $\|u_0\|_\infty$ in the right-hand side are also the same. The presence of the logarithmic corrections is the mark of the critical exponent.

Case 3. We now consider the case $q = p_c/2 = N/(N+1)$ and choose

$$\varrho(u) = u^{(N+1)/N} (\log M - \log u)^{(N+1)/N}, \; M = e\|u_0\|_\infty.$$

Then

$$\varrho'(u) = \frac{N+1}{N} u^{1/N} \left[(\log M - \log u)^{(N+1)/N} - (\log M - \log u)^{1/N} \right]$$

and

$$\varrho''(u) = \frac{N+1}{N^2} u^{-(N-1)/N} \left[\left(\log \frac{M}{u} \right)^{(N+1)/N} - (N+2) \left(\log \frac{M}{u} \right)^{1/N} + \left(\log \frac{M}{u} \right)^{-(N-1)/N} \right].$$

Thus, after straightforward calculations, we obtain

$$R_1 = \frac{N+1}{N} (\log M - \log u), \quad R_2 = \frac{N+1}{N} (\log M - \log u - 1).$$

Therefore

$$\begin{aligned}
R &= \frac{2(N-1)}{N}(\log M - \log u)w^{(p_c+2)/2} - \frac{2}{N}(\log M - \log u)w^{(q+2)/2} + \frac{2}{N}w^{(q+2)/2} \\
&\geq \frac{1}{N}(\log M - \log u)\left[2(N-1)w^{(p_c+2)/2} - 2w^{(q+2)/2}\right],
\end{aligned}$$

and

$$Lw := \partial_t w - Aw - B\cdot\nabla w + \frac{1}{N}(\log M - \log u)\left[2(N-1)w^{(p_c+2)/2} - 2w^{(q+2)/2}\right] \leq 0.$$

As a supersolution, we take

$$W(t) = \left(\frac{2}{N-1}\right)^{2(N+1)/N} + \left(\frac{N+1}{(N-1)t}\right)^{(N+1)/N}$$

and deduce, recalling that $N \geq 2$ and that $\log M - \log u \geq 1$:

$$\begin{aligned}
LW(t) = &-\left(\frac{N+1}{N-1}\right)^{(N+1)/N}\frac{N+1}{N}t^{-(2N+1)/N} + \frac{N-1}{N}(\log M - \log u)W(t)^{(2N+1)/(N+1)} \\
&+ \frac{1}{N}(\log M - \log u)W(t)^{(3N+2)/2(N+1)}\left((N-1)W(t)^{N/2(N+1)} - 2\right) \\
\geq &-\left(\frac{N+1}{N-1}\right)^{(N+1)/N}\frac{N+1}{N}t^{-(2N+1)/N} + \frac{N-1}{N}\left(\frac{N+1}{(N-1)t}\right)^{(2N+1)/N} = 0.
\end{aligned}$$

The comparison principle gives

$$|\nabla v(t,x)| \leq \left(\frac{2}{N-1}\right)^{(N+1)/N} + \left(\frac{N+1}{(N-1)t}\right)^{(N+1)/2N},$$

which implies (1.27).

Case 4. Finally, for $p = p_c$ and $q \in (0, p/2)$, we notice that $k + p_c - q - 1 = (p_c - 2q)/2 > 0$, hence we proceed as in Section 2.2. The estimates (1.21), (1.22) and (1.23) hold according to whether $q \in (p_c - 1, p_c/2)$, $q = p_c - 1$ or $q \in (0, p_c - 1)$. This ends the proof of Theorem 1.5.

2.4 Gradient estimates for $p < p_c$ and $q \geq 1 - k$

We want now to follow the same idea as in Section 2.1 and look for a function ϱ such that $R_1 = 1$, that is, ϱ is a solution of the following ordinary differential equation:

$$k(\varrho')^2 - \varrho\varrho'' = \varrho^{2-p}. \tag{2.18}$$

This equation can be reduced to a first order ordinary differential equation by using the standard trick of forcing the change of variable $\varrho' = f(\varrho)$, thus $\varrho'' = f(\varrho)f'(\varrho)$. Then $f(\varrho)$ solves the ordinary differential equation

$$f'(\varrho)f(\varrho) = \frac{k}{\varrho}f^2(\varrho) - \varrho^{1-p},$$

which can be explicitly integrated if we make a further change of variable by letting $f(\varrho) = \varrho^k g(\varrho)$. Then

$$g(\varrho)g'(\varrho) = -\varrho^{1-p-2k},$$

and, since $2-p-2k = (2-p)(N+1)(p_c-p)/2(p-1) > 0$, we find

$$g(\varrho) = \left(\frac{2(K_0^{2-p-2k} - \varrho(r)^{2-p-2k})}{2-p-2k} \right)^{1/2},$$

where K_0 is a generic constant. Coming back to the initial variable r, (2.18) transforms to

$$\varrho'(r) = f(\varrho(r)) = \varrho(r)^k \left(\frac{2(K_0^{2-p-2k} - \varrho(r)^{2-p-2k})}{2-p-2k} \right)^{1/2}. \tag{2.19}$$

In other words, ϱ is given in an implicit form through the integral expression

$$\left(\frac{2-p-2k}{2} \right)^{1/2} \int_0^{\varrho(r)} \frac{dz}{z^k (K_0^{2-p-2k} - z^{2-p-2k})^{1/2}} = r, \quad r \in [0, \|u_0\|_\infty].$$

Using the homogeneity of the integrand to scale K_0 out, we end up with

$$\left(\frac{(2-p-2k)K_0^p}{2} \right)^{1/2} \int_0^{\varrho(r)/K_0} \frac{dz}{z^k (1 - z^{2-p-2k})^{1/2}} = r.$$

A natural choice is then to take $\varrho(\|u_0\|_\infty) = K_0$ which leads to

$$\left(\frac{(2-p-2k)K_0^p}{2} \right)^{1/2} \int_0^{1} \frac{dz}{z^k (1 - z^{2-p-2k})^{1/2}} = \|u_0\|_\infty,$$

that is,

$$K_0 = \kappa \; \|u_0\|_\infty^{2/p} \tag{2.20}$$

for some positive constant κ depending only on N, p, and q. We also deduce from (2.19) that $\varrho'(r) \le C\varrho(r)^k K_0^{(2-p-2k)/2}$, hence, since $k<1$ and $\varrho(0)=0$, we find

$$\varrho(r) \le C K_0^{(2-p-2k)/2(1-k)} r^{1/(1-k)}, \quad r \in [0, \|u_0\|_\infty]. \tag{2.21}$$

We may now proceed along the lines of Section 2.1. Since $R_1 = 1$ by (2.18), it follows from (2.3) and (2.19) that

$$R = 2(p-1)w^{(p+2)/2} + 2(q-1)\varrho(u)^{q-1+k} \left(\frac{2(K_0^{2-p-2k} - \varrho(u)^{2-p-2k})}{2-p-2k} \right)^{1/2} w^{(q+2)/2}. \tag{2.22}$$

If $q \ge 1$ we omit the term coming from R_2 as it is non-negative and deduce from (2.22) and the comparison principle that

$$|\nabla u(t,x)| \le \varrho(u(t,x))(p(p-1)t)^{-1/p}, \quad (t,x) \in Q_\infty. \tag{2.23}$$

We plug the estimates (2.20) and (2.21) into (2.23) and obtain the following estimate

$$|\nabla u(t,x)| u(t,x)^{-1/(1-k)} \le C \|u_0\|_\infty^{(2-p-2k)/p(1-k)} t^{-1/p},$$

whence

$$|\nabla u^{-k/(1-k)}(t,x)| \le C \|u_0\|_\infty^{(2-p-2k)/p(1-k)} t^{-1/p}, \tag{2.24}$$

if $k \neq 0$ (that is, $p \neq p_{sc}$) and

$$|\nabla \log u(t,x)| \leq C\|u_0\|_\infty^{(2-p)/p} t^{-1/p}, \tag{2.25}$$

if $k = 0$, that is, $p = p_{sc} = 2(N+1)/(N+3)$.

We are left with the case $q \in [1-k, 1)$ (which is only possible if $k > 0$, thus $p > p_{sc}$). In this case, starting from (2.22), we use the monotonicity of ϱ, the identity (2.20) and (2.21), and compensate the negative term coming from R_2 in the following way:

$$\begin{aligned} R &\geq 2(p-1)w^{(p+2)/2} - 2(1-q)CK_0^{(2-p-2k)/2}\varrho(\|u_0\|_\infty)^{q-1+k}w^{(q+2)/2} \\ &\geq 2(p-1)w^{(p+2)/2} - C\|u_0\|_\infty^{((2-p-2k)+2(q-1+k))/p}w^{(q+2)/2} \\ &= 2(p-1)w^{(p+2)/2} - C\|u_0\|_\infty^{(2q-p)/p}w^{(q+2)/2}. \end{aligned}$$

Arguing as in Section 2.1, we conclude that

$$|\nabla u(t,x)| \leq C\varrho(u(t,x))\left(\|u_0\|_\infty^{(2q-p)/p(p-q)} + t^{-1/p}\right), \quad (t,x) \in Q_\infty.$$

Using again the estimates (2.20) and (2.21), we arrive to our final estimate

$$\left|\nabla u^{-k/(1-k)}(t,x)\right| \leq C\|u_0\|_\infty^{(2-p-2k)/p(1-k)}\left(\|u_0\|_\infty^{(2q-p)/p(p-q)} + t^{-1/p}\right) \tag{2.26}$$

for $(t,x) \in Q_\infty$.

2.5 Gradient estimates for the singular diffusion equation (1.3)

A careful look at the proofs of the gradient estimates (1.19), (1.25), (2.24), and (2.25) reveals that the contribution from the absorption term is always omitted so that these estimates are also true for solutions to the singular diffusion equation (1.3) with initial data satisfying (1.8). Since these gradient estimates seem to have been unnoticed before, we provide here a precise statement.

Theorem 2.2. *Consider a function u_0 satisfying* (1.8) *and let Φ be the solution to* (1.3) *with initial condition u_0. Then:*

(i) For $p \in (p_c, 2)$, we have

$$\left|\nabla \Phi^{-(2-p)/p}(t,x)\right| \leq \left(\frac{2-p}{p}\right)^{(p-1)/p} \eta^{1/p} t^{-1/p}, \ (t,x) \in Q_\infty.$$

(ii) For $p = p_c$, we have

$$\left|\nabla \Phi^{-1/N}(t,x)\right| \leq C\left(\log\left(\frac{e\|u_0\|_\infty}{\Phi(t,x)}\right)\right)^{1/p_c} t^{-1/p_c}, \ (t,x) \in Q_\infty.$$

(iii) For $p \in (p_{sc}, p_c)$, we have $k \in (0,1)$ and

$$\left|\nabla \Phi^{-k/(1-k)}(t,x)\right| \leq C\|u_0\|_\infty^{(2-p-2k)/p(1-k)} t^{-1/p}$$

for $(t,x) \in Q_\infty$ such that $\Phi(t,x) > 0$.

(iv) For $p = p_{sc}$*, we have* $k = 0$ *and*

$$|\nabla \log \Phi(t,x)| \le C\|u_0\|_\infty^{(2-p)/p} t^{-1/p}$$

for $(t,x) \in Q_\infty$ *such that* $\Phi(t,x) > 0$*.*

(v) For $p \in (1, p_{sc})$*, we have* $k < 0$ *and*

$$\left|\nabla \Phi^{|k|/(1+|k|)}(t,x)\right| \le C\|u_0\|_\infty^{(2-p-2k)/p(1-k)} t^{-1/p}, \ (t,x) \in Q_\infty.$$

2.6 A gradient estimate coming from the Hamilton-Jacobi term

Apart from the previous gradient estimates, which result either from the sole diffusion or are the outcome of the competition between the two terms, we can prove another one which is an extension of a known result for the non-diffusive Hamilton-Jacobi equation. We assume that $p > p_{sc} = 2(N+1)/(N+3)$, although in the applications we will only need the range $p \ge p_c$.

Case 1: $q < 1$. As in [14], take $\varphi(r) = \|u_0\|_\infty - r^2$ directly in (2.4) and (2.5). Then $v = (\|u_0\|_\infty - u)^{1/2}$, and

$$R_2 = -2^{q-1} v^{q-2}, \quad R_1 = 2^{p-2}(1+k)v^{p-4}.$$

Since we are in the range $q < 1$ and $p > p_{sc}$, we notice that $R_1 > 0$ and we can forget about the effect of this term. We deduce that

$$Lw := \partial_t w - Aw - B \cdot \nabla w + 2^q(1-q)(\|u_0\|_\infty - u)^{(q-2)/2} w^{(q+2)/2} \le 0.$$

We then notice that the function $W(t) = K\|u_0\|_\infty^{(2-q)/q} t^{-2/q}$, with a suitable choice of K, is a supersolution for the operator L, since

$$\begin{aligned} LW(t) &= \Big[2^q(1-q)K^{(q+2)/2}\,(\|u_0\|_\infty - u)^{(q-2)/2}\,\|u_0\|_\infty^{((2-q)/q)+((2-q)/2)} \\ &\qquad - \frac{2}{q}K\|u_0\|_\infty^{(2-q)/q}\Big]\, t^{-(q+2)/q} \\ &\ge 2K\left[2^{q-1}(1-q)K^{q/2} - \frac{1}{q}\right]\|u_0\|_\infty^{(2-q)/q} t^{-(q+2)/q} \ge 0 \end{aligned}$$

as soon as we choose $K^{q/2} = 2^{1-q}(1-q^2)$. By the comparison principle, we find that

$$\left|\nabla(\|u_0\|_\infty - u(t,x))^{1/2}\right| \le C\|u_0\|_\infty^{(2-q)/2q} t^{-1/q}.$$

Noticing that

$$2\left|\nabla(\|u_0\|_\infty - u(t,x))^{1/2}\right| = (\|u_0\|_\infty - u(t,x))^{-1/2}|\nabla u(t,x)| \ge \|u_0\|_\infty^{-1/2}|\nabla u(t,x)|,$$

we conclude that

$$|\nabla u(t,x)| \le C\|u_0\|_\infty^{1/q} t^{-1/q}, \ (t,x) \in Q_\infty.$$

Case 2: $q > 1$. In this case, let us take $\varrho(u) = u^{1/q}$ in (2.6) and (2.7), as in [3]. We calculate

$$R_1 = \frac{k+q-1}{q^2} u^{(p-2q)/q} > 0, \quad R_2 = \frac{1}{q}.$$

Since we want only an estimate coming from the absorption term, we omit R_1 and we have

$$Lw := \partial_t w - Aw - B \cdot \nabla w + \frac{2(q-1)}{q} w^{(2+q)/2} \leq 0.$$

We then notice that the function $W(t) = [(q-1)t]^{-2/q}$ is a supersolution for the operator L. By the comparison principle, we find that

$$|\nabla u(t,x)| \leq \varrho(u(t,x)) \left[\frac{1}{(q-1)t} \right]^{1/q},$$

or equivalently

$$\left| \nabla u^{(q-1)/q}(t,x) \right| \leq \frac{1}{q} (q-1)^{(q-1)/q} t^{-1/q}, \quad (t,x) \in Q_\infty.$$

Remark 2.3. *There is no gradient estimate produced by the Hamilton-Jacobi term for $q = 1$, since its contribution vanishes in* (2.3)*. This is in fact due to the lack of strict convexity (or concavity) of the euclidean norm.*

3 Decay estimates for integrable initial data

We devote this section to the proof of Theorem 1.1. These decay rates will be improved in Section 5 for $p > p_c$ and initial data which decay at infinity more rapidly than what is required by mere integrability.

Proposition 3.1. *Let u be a solution to* (1.1)-(1.2) *with an initial condition u_0 satisfying* (1.8)*. The following decay estimates hold:*
(i) If $p > p_c$ and $q > q_\star = p - N/(N+1)$, we have

$$\|u(t)\|_\infty \leq C \|u_0\|_1^{p\eta} t^{-N\eta}, \ t > 0, \tag{3.1}$$

where $\eta = 1/[N(p-2)+p]$.
(ii) If $p > p_c$ and $q \in (N/(N+1), q_\star]$, we have

$$\|u(t)\|_\infty \leq C \|u_0\|_1^{q\xi} t^{-N\xi}, \ t > 0, \tag{3.2}$$

where $\xi = 1/[q(N+1) - N]$.

Proof. Denoting the solution to (1.3) with initial condition u_0 by Φ, the comparison principle guarantees that $u \leq \Phi$ in Q_∞ and (3.1) readily follows from [16, Theorem 3]. Next, the proof of (3.2) for $q > 1$ and $q \in (N/(N+1), 1)$ relies on (1.29) and (1.28), respectively, and is the same as that of [2, Proposition 1.4] and [5, Theorem 1] to which we refer. For $q = 1$ we reproduce *verbatim* the proof in [5, Section 3]. □

Since (1.1) is an autonomous equation, a simple consequence of Proposition 3.1 is the following:

Corollary 3.2. *Let u be a solution of* (1.1)-(1.2) *with an initial condition u_0 satisfying* (1.8)*. For $p \in (p_c, 2)$ and $q \in (N/(N+1), q_\star]$, we have*

$$\|u(t)\|_\infty \leq C \ \|u(s)\|_1^{q\xi} \ (t-s)^{-N\xi}, \quad 0 \leq s < t. \tag{3.3}$$

For $q > q_\star$ we have

$$\|u(t)\|_\infty \leq C \ \|u(s)\|_1^{p\eta} \ (t-s)^{-N\eta}, \quad 0 \leq s < t. \tag{3.4}$$

We next turn to the case $p = p_c$ and first establish that the solutions to the singular diffusion equation (1.3) with non-negative integrable initial data decay exponentially for large times. Though this property is expected, a proof does not seem to be available in the literature.

Proposition 3.3. *Consider a function* u_0 *satisfying* (1.8) *and let* Φ *be the solution to* (1.3) *with initial condition* u_0 *and* $p = p_c$. *Then*

$$\|\Phi(t)\|_\infty \le C' \ \|u_0\|_\infty \ e^{-Ct/\|u_0\|_1^{2/(N+1)}} , \quad t \ge 0 . \tag{3.5}$$

Proof. By Theorem 2.2, we have

$$\begin{aligned} |\nabla\Phi(t,x)| &= N \ \Phi(t,x)^{(N+1)/N} \ \left|\nabla\Phi^{-1/N}(t,x)\right| \\ &\le C \ \Phi(t,x)^{(N+1)/N} \ \left(\log\left(\frac{e\|u_0\|_\infty}{\Phi(t,x)}\right)\right)^{1/p_c} \ t^{-1/p_c} \\ |\nabla\Phi(t,x)|^{p_c} &\le C \ \Phi(t,x)^2 \ \left(\log\left(\frac{e^{3/2}\|u_0\|_\infty}{\Phi(t,x)}\right)\right) \ t^{-1} . \end{aligned}$$

Noticing that the function $z \mapsto z^2 \log\left(e^{3/2}\|u_0\|_\infty/z\right)$ is non-decreasing in $[0, \|u_0\|_\infty]$ and that $0 \le \Phi \le \|u_0\|_\infty$ in Q_∞, we conclude that

$$|\nabla\Phi(t,x)|^{p_c} \le C \ \|\Phi(t)\|_\infty^2 \ \log\left(\frac{e^{3/2}\|u_0\|_\infty}{\|\Phi(t)\|_\infty}\right) \ t^{-1}$$

for $(t,x) \in Q_\infty$, while the Gagliardo-Nirenberg inequality

$$\|w\|_\infty \le C \ \|\nabla w\|_\infty^{N/(N+1)} \ \|w\|_1^{1/(N+1)} \quad \text{for} \quad w \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N) , \tag{3.6}$$

ensures that

$$\|\Phi(t)\|_\infty^2 \le C \ \|\nabla\Phi(t)\|_\infty^{p_c} \ \|\Phi(t)\|_1^{p_c/N} , \quad t > 0 .$$

Combining the above two inequalities with the conservation of mass $\|\Phi(t)\|_1 = \|u_0\|_1$ [16, Theorem 2], we end up with

$$\begin{aligned} \|\Phi(t)\|_\infty^2 &\le C\|\Phi(t)\|_\infty^2 \ \log\left(\frac{e^{3/2}\|u_0\|_\infty}{\|\Phi(t)\|_\infty}\right) \ \frac{\|u_0\|_1^{p_c/N}}{t} \\ e^{Ct/\|u_0\|_1^{p_c/N}} &\le \frac{e^{3/2}\|u_0\|_\infty}{\|\Phi(t)\|_\infty} , \end{aligned}$$

from which (3.5) follows. $\square$

Proof of Theorem 1.1. The estimates (1.9) and (1.10) are proved in Proposition 3.1. The exponential decay (1.11) follows from Proposition 3.3 and the comparison principle when $p = p_c$ while it is proved as in [5, Theorem 2] for $p > p_c$ and $q = N/(N+1)$, the main tool of the proof being the gradient estimate (1.28). For $p \ge p_c$ and $q \in (0, N/(N+1))$, the finite time extinction (1.12) is a feature of the absorption term and is also a consequence of (1.28). We refer to [5, Theorem 1] or [21, Theorem 3.1] for a proof. Finally, the extinction for $p \in (1, p_c)$ follows by comparison with the singular diffusion equation (1.3) for which finite time extinction is known to occur for initial data in $L^r(\mathbb{R}^N)$ with suitable r [11, 16, 25], noting that $L^1 \cap L^\infty \subset L^r$ for any $r \in (1, \infty)$. $\square$

4 Large time behavior of $\|u(t)\|_1$

In this section we study the possible values of the limit as $t \to \infty$ of the L^1-norm of solutions u to (1.1)-(1.2) with initial data u_0 satisfying (1.8). The case $p \in (1, p_c)$ being obvious as u vanishes identically after a finite time by Theorem 1.1, we assume in this section that $p \geq p_c$ and first state the time monotonicity of the L^1-norm of u

$$\|u(t)\|_1 \leq \|u(s)\|_1 \leq \|u_0\|_1 \,, \qquad t > s \geq 0 \,, \tag{4.1}$$

which follows by construction of the solution, see (6.3) below. This last inequality can actually be improved to an equality for $p \geq p_c$ as we shall see now.

Proposition 4.1. *If $p \in (p_c, 2)$, $q \in [p/2, \infty)$, and u_0 satisfies* (1.8)*, then*

$$\|u(t)\|_1 + \int_0^t \int |\nabla u(s,x)|^q \; dxds = \|u_0\|_1 \,, \qquad t \geq 0. \tag{4.2}$$

Remark 4.2. *Let us point out here that this result is not obvious as it is clearly false for the singular diffusion equation* (1.3) *for $p < p_c$ for which we have extinction in finite time. Therefore, it may only hold true for $p \geq p_c$ and we refer to [16, Theorem 2] for a proof for* (1.3)*. The proof of Proposition 4.1 given below for $p > p_c$ (and $q \geq p/2$) is however of a completely different nature, relying on the gradient estimates* (1.19) *and* (1.20)*, and provides an alternative proof of the mass conservation for* (1.3) *for $p > p_c$. The case $p = p_c$ will be considered in the next proposition, the proof relying on arguments from [16].*

Proof. Let ϑ be a non-negative and smooth compactly supported function in $\mathbb{R}^N$ such that $0 \leq \vartheta \leq 1$, $\vartheta(x) = 1$ for $x \in B_1(0)$ and $\vartheta(x) = 0$ for $x \in \mathbb{R}^N \setminus B_2(0)$. For $R > 1$ and $x \in \mathbb{R}^N$, we define $\vartheta_R(x) := \vartheta(x/R)$. Since $p/(2-p) > 1$, the function $\vartheta_R^{p/(2-p)}$ is a non-negative compactly supported C^1-smooth function and it follows from (6.2) that, for $t > 0$,

$$\begin{aligned} I_R(t) &:= \int \vartheta_R^{p/(2-p)} \; u(t) \; dx + \int_0^t \int \vartheta_R^{p/(2-p)} \; |\nabla u(s)|^q \; dxds \\ &= \int \vartheta_R^{p/(2-p)} \; u_0 \; dx - \int_0^t \int \nabla \left(\vartheta_R^{p/(2-p)} \right) \cdot \left(|\nabla u|^{p-2} \; \nabla u \right)(s) \; dxds \,. \end{aligned} \tag{4.3}$$

On the one hand, since $u_0 \in L^1(\mathbb{R}^N)$ and $\vartheta_R^{p/(2-p)} \longrightarrow 1$ as $R \to \infty$ with $\left| \vartheta_R^{p/(2-p)} \right| \leq 1$, the Lebesgue dominated convergence theorem guarantees that

$$\lim_{R \to \infty} \int \vartheta_R(x)^{p/(2-p)} \; u_0(x) \; dx = \|u_0\|_1 \,. \tag{4.4}$$

On the other hand, since $p > p_c$ and $q \geq p/2$, u satisfies the gradient estimate

$$\left| \nabla u^{(p-2)/p}(s,x) \right| \leq C(u_0) \; \left(1 + s^{-1/p} \right) \,, \qquad (s,x) \in Q_\infty \,,$$

by (1.19) and (1.20). Since $|\nabla u| = (p/(2-p)) \; u^{2/p} \; \left| \nabla u^{(p-2)/p} \right|$ and $2(p-1) < p$, we infer

from the previous gradient estimate and Hölder's and Young's inequalities that

$$
\begin{aligned}
& \left| \int_0^t \int \nabla \left(\vartheta_R^{p/(2-p)} \right) (x) \cdot \left(|\nabla u|^{p-2}\ \nabla u \right) (s,x)\ dxds \right| \\
\leq & \ C \int_0^t \int \vartheta_R(x)^{2(p-1)/(2-p)}\ |\nabla \vartheta_R(x)|\ u(s,x)^{2(p-1)/p}\ \left| \nabla u^{(p-2)/p}(s,x) \right|^{p-1}\ dxds \\
\leq & \ C(u_0) \int_0^t \left(1 + s^{-(p-1)/p} \right)\ \|\nabla \vartheta_R\|_{p/(2-p)}\ \left(\int \vartheta_R(x)^{p/(2-p)}\ u(s,x)\ dx \right)^{2(p-1)/p}\ ds \\
\leq & \ C(u_0,\vartheta)\ R^{-(N+1)(p-p_c)/p} \int_0^t \left(1 + s^{-(p-1)/p} \right)\ I_R(s)^{2(p-1)/p}\ ds \\
\leq & \ C(u_0,\vartheta)\ R^{-(N+1)(p-p_c)/p} \int_0^t \left(1 + s^{-(p-1)/p} \right)\ (1 + I_R(s))\ ds\,.
\end{aligned}
\tag{4.5}
$$

It now follows from (4.3), (4.5), and Gronwall's lemma that

$$
I_R(t) \leq (1 + \|u_0\|_1)\ \exp\left\{ C(u_0,\vartheta)\ R^{-(N+1)(p-p_c)/p}\ (t + t^{1/p}) \right\} - 1\,, \qquad t \geq 0\,. \tag{4.6}
$$

Since $\vartheta_R^{p/(2-p)} \longrightarrow 1$ as $R \to \infty$ and the right-hand side of (4.6) is bounded independently of $R > 1$, we deduce from (4.6) and Fatou's lemma that $u(t) \in L^1(\mathbb{R}^N)$ and $|\nabla u|^q \in L^1((0,t) \times \mathbb{R}^N)$ for every $t > 0$. We are then in a position to apply once more the Lebesgue dominated convergence theorem to conclude that

$$
\lim_{R\to\infty} I_R(t) = \|u(t)\|_1 + \int_0^t \int |\nabla u(s,x)|^q\ dxds\,, \qquad t > 0\,, \tag{4.7}
$$

while (4.5), (4.6), and the assumption $p > p_c$ ensure that

$$
\lim_{R\to\infty} \int_0^t \int \nabla \left(\vartheta_R^{p/(2-p)} \right) (x) \cdot \left(|\nabla u|^{p-2}\ \nabla u \right) (s,x)\ dxds = 0\,, \qquad t > 0\,. \tag{4.8}
$$

We may then pass to the limit as $R \to \infty$ in (4.3) and use (4.4), (4.7), and (4.8) to obtain (4.2). □

We complete now the panorama with the corresponding result for $p = p_c > 1$, which requires $N \geq 2$.

Proposition 4.3. *If $p = p_c$, $q > 0$, and u_0 satisfies* (1.8) *along with*

$$
u_0(x) \leq C_0\ |x|^{-N}\,, \quad x \in \mathbb{R}^N\,, \tag{4.9}
$$

for some $C_0 > 0$, then

$$
\|u(t)\|_1 + \int_0^t \int |\nabla u(s,x)|^q\ dxds = \|u_0\|_1\,, \qquad t \geq 0. \tag{4.10}
$$

Proof. A straightforward computation shows that $(t,x) \mapsto C_0\ |x|^{-N}$ is a supersolution to (1.1) in $(0,\infty) \times \mathbb{R}^N \setminus \{0\}$ and we infer from (4.9) and the comparison principle that

$$
u(t,x) \leq C_0\ |x|^{-N}\,, \quad (t,x) \in [0,\infty) \times \mathbb{R}^N \setminus \{0\}\,. \tag{4.11}
$$

Next, let ϑ be a non-negative and smooth compactly supported function in $\mathbb{R}^N$ such that $0 \le \vartheta \le 1$, $\vartheta(x) = 1$ for $x \in B_1(0)$, and $\vartheta(x) = 0$ for $x \in \mathbb{R}^N \setminus B_2(0)$. For $R > 1$ and $x \in \mathbb{R}^N$, we define $\vartheta_R(x) := \vartheta(x/R)$. We multiply (1.1) by $(1-\vartheta_R)^{p_c}\ u$, integrate over $\mathbb{R}^N$, and use Young's inequality to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int (1-\vartheta_R)^{p_c}\ u^2\ dx &\le -\int \nabla\left((1-\vartheta_R)^{p_c}\ u\right)\cdot |\nabla u|^{p_c-2}\ \nabla u\ dx \\
&\le -\int (1-\vartheta_R)^{p_c}\ |\nabla u|^{p_c}\ dx + p_c\ \int |\nabla\vartheta_R|\ \left((1-\vartheta_R)\ |\nabla u|\right)^{p_c-1}\ u\ dx \\
&\le -(2-p_c)\ \int (1-\vartheta_R)^{p_c}\ |\nabla u|^{p_c}\ dx + \int |\nabla\vartheta_R|^{p_c}\ u^{p_c}\ dx\,.
\end{aligned}$$

Integrating with respect to time over $(0,t)$ and using the properties of ϑ_R, (4.9), and (4.11) give

$$\begin{aligned}
(2-p_c)\ \int_0^t \int_{\{|x|\ge 2R\}} |\nabla u|^{p_c}\ dxds &\le (2-p_c)\ \int_0^t \int (1-\vartheta_R)^{p_c}\ |\nabla u|^{p_c}\ dxds \\
&\le \frac{1}{2}\ \int (1-\vartheta_R)^{p_c}\ u_0^2\ dx + \frac{1}{R^{p_c}}\ \int_0^t \int \left|\nabla\vartheta\left(\frac{x}{R}\right)\right|^{p_c}\ u^{p_c}\ dxds \\
&\le \frac{C_0}{2}\ \int_{\{|x|\ge R\}} \frac{u_0(x)}{|x|^N}\ dx + \frac{C_0^{p_c-1}\|\nabla\vartheta\|_\infty^{p_c}}{R^{p_c}}\ \int_0^t \int_{\{|x|\ge R\}} \frac{u(s,x)}{|x|^{N(p_c-1)}}\ dxds \\
&\le \frac{C_0}{2R^N}\ \int_{\{|x|\ge R\}} u_0(x)\ dx + \frac{C_0^{p_c-1}\|\nabla\vartheta\|_\infty^{p_c}}{R^N}\ \int_0^t \int_{\{|x|\ge R\}} u(s,x)\ dxds\,,
\end{aligned}$$

whence

$$\int_0^t \int_{\{|x|\ge 2R\}} |\nabla u|^{p_c}\ dxds \le \frac{C(\vartheta,u_0)}{R^N}\ \omega(t,R)\,, \tag{4.12}$$

with

$$\omega(t,R) := \int_{\{|x|\ge R\}} u_0(x)\ dx + \int_0^t \int_{\{|x|\ge R\}} u(s,x)\ dxds\,.$$

Now, owing to (4.12) and Hölder's inequality, we have

$$\begin{aligned}
&\left|\int_0^t \int \nabla\left(\vartheta_R^N\right)\cdot |\nabla u|^{p_c-2}\nabla u\ dxds\right| \\
\le&\ N\ \left(\int_0^t \int_{\{|x|\ge R\}} |\nabla u|^{p_c}\ dxds\right)^{(p_c-1)/p_c}\ \left(\int_0^t \int |\nabla\vartheta_R|^{p_c}\ dxds\right)^{1/p_c} \\
\le&\ N\ \left[\frac{2^N C(\vartheta,u_0)}{R^N}\ \omega\left(t,\frac{R}{2}\right)\right]^{(N-1)/2N}\ \|\nabla\vartheta\|_{p_c}\ t^{1/p_c}\ R^{(N-p_c)/p_c} \\
\le&\ C(\vartheta,u_0)\ t^{1/p_c}\ \omega\left(t,\frac{R}{2}\right)^{(N-1)/2N}\,.
\end{aligned}$$

Since $u \in L^\infty(0,t;L^1(\mathbb{R}^N))$ by (6.3) and $u_0 \in L^1(\mathbb{R}^N)$, it readily follows from the Lebesgue dominated convergence theorem that $\omega(t,R/2) \to 0$ as $R \to \infty$. We have thus proved that (4.8) also holds true for $p = p_c$ (since $p_c/(2-p_c) = N$) and we can proceed as in the end of the proof of Proposition 4.1 to complete the proof. $\square$

We prove now a first result concerning non-extinction in finite time in the range $q > p/2$. Apart from the interest by itself, this result is also a technical step in the proof of the next estimates.

Proposition 4.4. *Let $p \geq p_c$, $q \in (p/2, \infty)$, and an initial condition u_0 satisfying* (1.8) *as well as* (4.9) *if $p = p_c$. Then the solution of* (1.1)-(1.2) *cannot vanish in finite time.*

Proof. We borrow some ideas from [1, Lemma 4.1]. Assume for contradiction that there exists $T \in (0, \infty)$ such that $u(T) \equiv 0$ and $\|u(t)\|_1 > 0$ for any $t \in [0, T)$. For $\theta \in (0, 1)$ to be specified later, define

$$E_\theta(t) = \int u(t,x)^{1+\theta}\,dx, \quad t \geq 0. \tag{4.13}$$

Let $\lambda > 0$ (to be chosen later) and $Q \in (p/2, p)$ such that $Q \leq q$. We use Proposition 4.1 for $p > p_c$ or Proposition 4.3 for $p = p_c$, (2.8), and Hölder's inequality to get

$$\begin{aligned} \frac{d}{dt}\|u(t)\|_1 &= -\int |\nabla u|^q\,dx \geq -\|\nabla u_0\|_\infty^{q-Q} \int |\nabla u|^Q u^{-\lambda} u^\lambda\,dx \\ &\geq -C(u_0) \left(\int |\nabla u|^p u^{-p\lambda/Q}\,dx \right)^{Q/p} \left(\int u^{p\lambda/(p-Q)}\,dx \right)^{(p-Q)/p}. \end{aligned}$$

We now choose λ in order to find the derivative of E_θ in the first factor in the right-hand side of the above inequality. More specifically, by differentiating in (4.13) and using (1.1), we find

$$\begin{aligned} \frac{d}{dt}E_\theta(t) &= (1+\theta) \int u(t,x)^\theta (\Delta_p u(t,x) - |\nabla u(t,x)|^q)\,dx \\ &\leq -\theta(1+\theta) \int u(t,x)^{\theta-1} |\nabla u(t,x)|^p\,dx, \end{aligned}$$

hence, we choose λ such that $p\lambda/Q = 1 - \theta > 0$. The inequality thus becomes

$$\frac{d}{dt}\|u(t)\|_1 \geq -C(u_0,\theta) \left(-\frac{d}{dt}E_\theta(t) \right)^{Q/p} \left(\int u(t,x)^{Q(1-\theta)/(p-Q)}\,dx \right)^{(p-Q)/p}. \tag{4.14}$$

We choose θ such that $Q(1-\theta)/(p-Q) = 1$, that is $\theta = (2Q-p)/Q \in (0,1)$. Using Young's inequality, we arrive to the differential inequality

$$\frac{d}{dt}\|u(t)\|_1 \geq -C(u_0,\theta) \left(-\frac{d}{dt}E_\theta(t) \right)^{Q/p} \|u(t)\|_1^{(p-Q)/p} \geq \varepsilon \frac{d}{dt}E_\theta(t) - C(u_0,\theta,\varepsilon)\|u(t)\|_1,$$

for $\varepsilon > 0$; we integrate it on (t, T) and use the time monotonicity (6.3) of $\|u\|_1$ to get

$$-\|u(t)\|_1 + C(u_0,\theta,\varepsilon)(T-t)\|u(t)\|_1 \geq -\|u(t)\|_1 + C(u_0,\theta,\varepsilon) \int_t^T \|u(s)\|_1\,ds \geq -\varepsilon E_\theta(t)\,,$$

whence

$$\liminf_{t\to T} \frac{E_\theta(t)}{\|u(t)\|_1} \geq \frac{1}{\varepsilon}\,. \tag{4.15}$$

But on the other hand, we notice that

$$\frac{E_\theta(t)}{\|u(t)\|_1} \leq \|u(t)\|_\infty^\theta \to 0 \quad \text{as } t \to T,$$

which is a contradiction with (4.15). Thus, there cannot be a finite extinction time $T > 0$. ∎

As a consequence of this non-extinction result, we are able to prove that, for $p > p_c$ and $q > p/2$, the positivity set is the whole set Q_∞.

Corollary 4.5. *If $p \ge p_c$, $q > p/2$, and u_0 satisfies* (1.8) *as well as* (4.9) *if $p = p_c$, then the solution of* (1.1)-(1.2) *is such that $u(t,x) > 0$ for $(t,x) \in Q_\infty$.*

Proof. We first consider the case $p > p_c$. Let $t > 0$ and $\delta \in (0,1)$. We first recall that, since $p > p_c$ and $q > p/2$, we have

$$\left|\nabla (u+\delta)^{(p-2)/p}(t,x)\right| \le \phi(t) := C(u_0)\ \left(1 + t^{-1/p}\right)\,, \quad x \in \mathbb{R}^N\,, \tag{4.16}$$

by (1.19) and (1.20), taking into account Remark 1.4 and (1.24). Fix $x_0 \in \mathbb{R}^N$. For $x \in \mathbb{R}^N$, we infer from (4.16) that

$$(u(t,x_0)+\delta)^{(p-2)/p} \le (u(t,x)+\delta)^{(p-2)/p} + \phi(t)\ |x-x_0|\,.$$

Multiplying the above inequality by $(u(t,x)+\delta)^{2/p}$ and integrating with respect to x over $B_r(x_0)$ for some $r > 0$ to be determined later give

$$\begin{aligned} & \left(\int_{B_r(x_0)} (u(t,x)+\delta)^{2/p}\ dx\right)\ (u(t,x_0)+\delta)^{(p-2)/p} \\ \le\ & \int_{B_r(x_0)} \left[u(t,x)+\delta+\phi(t)\ |x-x_0|\ (u(t,x)+\delta))^{2/p}\right]\ dx\,. \end{aligned}$$

Noting that

$$M(r,\delta) := \int_{B_r(x_0)} (u(t,x)+\delta)\ dx \le \left(\int_{B_r(x_0)} (u(t,x)+\delta)^{2/p}\ dx\right)^{p/2}\ |B_r(x_0)|^{(2-p)/2}$$

by Hölder's inequality, we obtain

$$|B_r(x_0)|^{(p-2)/p}\ M(r,\delta)^{2/p}\ (u(t,x_0)+\delta)^{(p-2)/p} \le M(r,\delta)\ \left(1 + r\ \phi(t)\ \|u(t)+\delta\|_\infty^{(2-p)/p}\right)\,,$$

$$|B_r(x_0)|^{(p-2)/p}\ M(r,\delta)^{(2-p)/p} \le (u(t,x_0)+\delta)^{(2-p)/p}\ \left(1 + r\ \phi(t)\ \|u(t)+\delta\|_\infty^{(2-p)/p}\right)\,.$$

Letting $\delta \to 0$, we end up with

$$|B_r(x_0)|^{-1}\ M(r,0) \le u(t,x_0)\ \left(1 + r\ \phi(t)\ \|u(t)\|_\infty^{(2-p)/p}\right)^{p/(2-p)}\,.$$

Since $M(r,0) \to \|u(t)\|_1$ as $r \to \infty$ and $\|u(t)\|_1 > 0$ by Proposition 4.4, we may fix r_0 large enough such that $M(r_0,0) > 0$ and deduce from the above inequality with $r = r_0$ that

$$0 < |B_{r_0}(x_0)|^{-1}\ M(r_0,0) \le u(t,x_0)\ \left(1 + r_0\ \phi(t)\ \|u(t)\|_\infty^{(2-p)/p}\right)\,,$$

which shows the positivity of $u(t,x_0)$.

Next, if $p = p_c$, $q \in (p_c/2,\infty)$, $\delta \in (0,1)$, and $(t,x) \in Q_\infty$, it follows from (1.25), (1.26), and Remark 1.6 that

$$\left|\nabla (u+\delta)^{-1/N}(t,x)\right| \le C(u_0)\ \left(\log\left(\frac{e\|u_0\|_\infty}{u(t,x)+\delta}\right)\right)^{1/p_c}\ \left(1 + t^{-1/p_c}\right)\,.$$

Fix $\theta \in (0,1/N)$. Then, owing to the boundedness of the function $r \mapsto r^{(1-N\theta)/N}|\log r|^{1/p_c}$ for $r \in [0, \|u_0\|_\infty + 1]$, we have

$$\begin{aligned}\left|\nabla(u+\delta)^{-\theta}(t,x)\right| &= N\theta\ (u(t,x)+\delta)^{(1-N\theta)/N}\ \left|\nabla(u+\delta)^{-1/N}(t,x)\right| \\ &\leq C(\theta,u_0)\ \left(1+t^{-1/p_c}\right),\end{aligned}$$

for $(t,x) \in Q_\infty$ and we may proceed as in the previous case to establish the claimed positivity of u in Q_∞. □

We are now in a position to prove the two main results of this section.

Proposition 4.6. *Let u be a solution to* (1.1)-(1.2) *with an initial condition u_0 satisfying* (1.8) *as well as* (4.9) *if $p = p_c$. If $p \geq p_c$ and $q > q_\star$, then we have* $\lim_{t\to\infty} \|u(t)\|_1 > 0$.

Proof. From Proposition 4.1 (if $p > p_c$) and Proposition 4.3 (if $p = p_c$), we have, for any $1 \leq s \leq t < \infty$:

$$\|u(s)\|_1 = \|u(t)\|_1 + \int_s^t \int \left(u(\tau,x)^{-1/q}|\nabla u(\tau,x)|\right)^q u(\tau,x)\,dx\,d\tau\,. \tag{4.17}$$

We want to use the gradient estimates (1.19), (1.20), (1.25), and (1.26), and thus split the proof into three cases.

Case 1: $p > p_c$ **and** $q \geq 1$. In this case, by using the gradient estimate (1.19), together with the decay estimate of the L^∞-norm (3.1), we write, since $q > p/2$:

$$\begin{aligned}u(\tau,x)^{-1/q}\,|\nabla u(\tau,x)| &= C\ u(\tau,x)^{(2q-p)/pq}\left|\nabla u^{-(2-p)/p}(\tau,x)\right| \\ &\leq C\ \|u(\tau)\|_\infty^{(2q-p)/pq}\ \tau^{-1/p} \\ &\leq C\ \|u_0\|_1^{(2q-p)\eta/q}\ \tau^{-N\eta(2q-p)/pq-1/p},\end{aligned}$$

hence

$$\left(u^{-1/q}(\tau,x)\,|\nabla u(\tau,x)|\right)^q \leq C(u_0)\tau^{-\eta/\xi}.$$

Plugging this inequality into (4.17) and taking into account that $\xi < \eta$, it follows that

$$\begin{aligned}\|u(s)\|_1 &\leq \|u(t)\|_1 + C(u_0)\int_s^t \|u(\tau)\|_1\tau^{-\eta/\xi}\,d\tau \\ &\leq \|u(t)\|_1 + C(u_0)\|u(s)\|_1 s^{-\eta(N+1)(q-q_\star)},\end{aligned}$$

where we have used the time monotonicity (6.3) of the L^1-norm. We can rewrite the last inequality as

$$\|u(t)\|_1 \geq \|u(s)\|_1\left(1 - C(u_0)s^{-\eta(N+1)(q-q_\star)}\right). \tag{4.18}$$

Using again that the exponent of s in the right-hand side of (4.18) is negative, we realize that

$$\|u(t)\|_1 \geq \frac{1}{2}\|u(s)\|_1\,, \quad t \geq s\,,$$

for s large enough. Thus, using the non-extinction result of Proposition 4.4, we find that $\lim_{t\to\infty}\|u(t)\|_1 > 0$.

Case 2: $p > p_c$ **and** $q_\star < q < 1$. We use the same ideas as above, but with slight changes since the gradient estimate has now an extra term. Since (1.1) is autonomous, we infer from (1.20) and (3.1) that

$$\begin{aligned}
\left|\nabla u^{-(2-p)/p}(\tau,x)\right| &\le C\left(\left\|u\left(\frac{\tau}{2}\right)\right\|_\infty^{(2q-p)/p(p-q)} + \left(\frac{2}{\tau}\right)^{1/p}\right) \\
&\le C(u_0)\tau^{-1/p}\left(1+\tau^{-\eta(N+1)(q-q_\star)/(p-q)}\right) \le C(u_0)\tau^{-1/p},
\end{aligned}$$

for any $\tau \ge 1$. The proof then is the same as in Case 1 above.

Case 3: $p = p_c$ **and** $q > q_\star = p_c/2$. To estimate $u^{-1/q}\,|\nabla u|$, we use (1.11), (1.25) (if $q \ge 1$) or (1.26) (if $q \in (p_c/2, 1)$), and the boundedness of the function $z \mapsto z^{(2q-p_c)/2p_c q}\,\log(e\|u_0\|_\infty/z)$ in $[0, \|u_0\|_\infty]$ to obtain, since $\tau \ge s \ge 1$,

$$\begin{aligned}
u(\tau,x)^{-1/q}\,|\nabla u(\tau,x)| &\le C\ u(\tau,x)^{(2q-p_c)/p_c q}\,\left|\nabla u^{-1/N}(\tau,x)\right| \\
&\le C(u_0)\ u(\tau,x)^{(2q-p_c)/p_c q}\left(\log\left(\frac{e\|u_0\|_\infty}{u(\tau,x)}\right)\right)^{1/p_c}\ \tau^{-1/p_c} \\
&\le C(u_0)\ u(\tau,x)^{(2q-p_c)/2p_c q}\ \tau^{-1/p_c} \\
&\le C'(u_0)\ e^{-C(u_0)\tau}\,.
\end{aligned}$$

This estimate, (4.17), and the time monotonicity (6.3) of the L^1-norm lead us to

$$\|u(s)\|_1 \le \|u(t)\|_1 + C'(u_0)\ \|u(s)\|_1\ e^{-C(u_0)s}\,, \quad t \ge s\,,$$

and we complete the proof as above with the help of Proposition 4.4. $\square$

For the complementary case, things are different.

Proposition 4.7. *Let* $p \in (1,2)$ *and* $q \in (0, q_\star]$. *Then* $\lim_{t\to\infty} \|u(t)\|_1 = 0$.

Proof. The proof follows that of [2, Proposition 5.1]. For $t \ge 0$, we have

$$\|u(t)\|_1 + \int_0^t \int |\nabla u(s,x)|^q\,dxds \le \|u_0\|_1$$

by (6.3), hence $|\nabla u|^q \in L^1((0,\infty)\times\mathbb{R}^N)$. Therefore

$$\omega(t) := \int_t^\infty \int |\nabla u(s,x)|^q\,dxds \to 0 \text{ as } t \to \infty. \tag{4.19}$$

Consider now a non-negative and smooth compactly supported function ϑ such that $0 \le \vartheta \le 1$, $\vartheta(x) = 1$ for $x \in B_1(0)$ and $\vartheta(x) = 0$ for $x \in \mathbb{R}^N \setminus B_2(0)$ and define $\vartheta_R(x) = \vartheta(x/R)$ for $R > 1$ and $x \in \mathbb{R}^N$. We multiply the equation (1.1) by $1 - \vartheta_R$ and integrate over $(t_1, t_2) \times \mathbb{R}^N$ to obtain

$$\begin{aligned}
\int u(t_2,x)(1-\vartheta_R(x))\,dx &\le \int u(t_1,x)(1-\vartheta_R(x))\,dx \\
&\quad + \int_{t_1}^{t_2}\int |\nabla u(s,x)|^{p-2}\nabla u(s,x)\cdot\nabla\vartheta_R(x)\,dxds,
\end{aligned}$$

hence, taking into account the definition of ϑ_R,

$$\int\limits_{|x|\geq 2R} u(t_2,x)\,dx \leq \int\limits_{|x|\geq R} u(t_1,x)\,dx + \frac{1}{R}\int_{t_1}^{t_2}\int |\nabla u(s,x)|^{p-1}|\nabla\vartheta(x/R)|\,dxds. \tag{4.20}$$

We now divide the proof into two cases.

Case 1: $p \geq p_c$, $q \in [N/(N+1), q_\star]$. Let us first consider the case where $q \in [p-1, q_\star]$ and $q > N/(N+1)$. We apply Hölder's inequality to estimate

$$\begin{aligned} & \frac{1}{R}\int_{t_1}^{t_2}\int |\nabla u(s,x)|^{p-1}|\nabla\vartheta(x/R)|\,dxds \\ \leq\ & R^{(N(q-p+1)-q)/q}\|\nabla\vartheta\|_{q/(q-p+1)}(t_2-t_1)^{(q-p+1)/q}\left(\int_{t_1}^{t_2}\int |\nabla u(s,x)|^q\,dxds\right)^{(p-1)/q} \\ \leq\ & C(\vartheta)R^{(N(q-p+1)-q)/q}(t_2-t_1)^{(q-p+1)/q}\omega(t_1)^{(p-1)/q}, \end{aligned}$$

hence, replacing in (4.20) we obtain

$$\begin{aligned} \|u(t_2)\|_1 &= \int\limits_{|x|<2R} u(t_2,x)\,dx + \int\limits_{|x|\geq 2R} u(t_2,x)\,dx \\ &\leq CR^N\|u(t_2)\|_\infty + C(\vartheta)R^{(N(q-p+1)-q)/q}(t_2-t_1)^{(q-p+1)/q}\omega(t_1)^{(p-1)/q} \\ &\quad + \int\limits_{|x|\geq R} u(t_1,x)\,dx. \end{aligned} \tag{4.21}$$

Taking into account that $\|u(t_2)\|_\infty \leq C(u_0)(t_2-t_1)^{-N\xi}$ by (3.2), we optimize in R in the previous inequality. Choosing

$$R = R(t_1,t_2) := \omega(t_1)^{(p-1)/(N(p-1)+q)}(t_2-t_1)^{(qN\xi+q-p+1)/(q+N(p-1))},$$

we obtain

$$\begin{aligned} \|u(t_2)\|_\infty \leq & C(u_0,\vartheta)\ \omega(t_1)^{N(p-1)/(N(p-1)+q)}\ (t_2-t_1)^{qN(N+1)\xi(q-q_\star)/(N(p-1)+q)} \\ & + \int\limits_{|x|\geq R(t_1,t_2)} u(t_1,x)\ dx. \end{aligned}$$

Noting that

$$qN\xi + q - p + 1 = \xi\ (q(N+1)(q-p+1) + N(p-1)) > 0$$

since $\xi > 0$ and $q \geq p-1$, we may let $t_2 \to \infty$ in the previous estimate to obtain that $\|u(t_2)\|_\infty \to 0$ as $t_2 \to \infty$ when $q < q_\star$, and that

$$\lim_{t\to\infty}\|u(t)\|_1 \leq C(u_0,\vartheta)\omega(t_1)^{N(p-1)/(q+N(p-1))} \to 0 \text{ as } t_1 \to 0,$$

for $q = q_\star$.

In the remaining case we can always fix $Q \geq q$ such that $Q \in (p-1, q_\star)$ and $Q > N/(N+1)$. Introducing

$$\tilde{u}(t,x) := \|\nabla u_0\|_\infty^{-(Q-q)/(Q-p+1)}\ u\left(\|\nabla u_0\|_\infty^{((2-p)(Q-q))/(Q-p+1)}\ t, x\right), \qquad (t,x) \in Q_\infty,$$

we deduce from (1.1),(1.2), and (2.8) that

$$\begin{aligned}
\partial_t \tilde{u}(t,x) &= \|\nabla u_0\|_\infty^{-((p-1)(Q-q))/(Q-p+1)}\ \partial_t u\left(\|\nabla u_0\|_\infty^{((2-p)(Q-q))/(Q-p+1)}\ t,x\right)\\
&= \|\nabla u_0\|_\infty^{-((p-1)(Q-q))/(Q-p+1)}\ (\Delta_p u - |\nabla u|^q)\left(\|\nabla u_0\|_\infty^{((2-p)(Q-q))/(Q-p+1)}\ t,x\right)\\
&\le \Delta_p \tilde{u}(t,x) - \|\nabla u_0\|_\infty^{Q-q}\ |\nabla \tilde{u}(t,x)|^Q \left\|\nabla u\left(\|\nabla u_0\|_\infty^{((2-p)(Q-q))/(Q-p+1)}\ t\right)\right\|_\infty^{q-Q}\\
&\le \Delta_p \tilde{u}(t,x) - |\nabla \tilde{u}(t,x)|^Q\,,
\end{aligned}$$

with $\tilde{u}(0) = U_0 := \|\nabla u_0\|_\infty^{-(Q-q)/(Q-p+1)}\ u_0$. Denoting the solution to (1.1)-(1.2) with Q instead of q and U_0 instead of u_0 by U, the comparison principle entails that $\tilde{u} \le U$ in Q_∞. According to the choice of Q, we are in the situation of the previous case and thus $\|U(t)\|_1 \to 0$ as $t \to \infty$ and so do $\|\tilde{u}(t)\|_1$ and $\|u(t)\|_1$.

Case 2: $p \ge p_c$ **and** $q \in (0, N/(N+1))$ **or** $p < p_c$. It is an obvious consequence of the extinction in finite time established in Theorem 1.1. □

5 Improved decay rates and extinction

While the behavior of solutions u to (1.1) depends strongly on the values of p and q as depicted in Theorem 1.2, it turns out that, as we shall see below, the proofs also vary with these two parameters. Indeed, recalling the definition of q_1 in (1.7), finite time extinction will follow by the comparison principle when either $p \in (1, p_c)$ or $p \ge p_c$ and $q \in (0, q_1]$, while a differential inequality will be used for $p > p_c$ and $q \in (q_1, p/2)$. A similar differential inequality will actually allow us to prove the stated temporal decay rates for $p > p_c$ and $q \in [p/2, q_\star)$. The particular case $p = p_c$ has to be handled separately. Still, the proof of Theorem 1.2 for $p \in (p_c, 2)$ and $q \in (q_1, q_\star)$, $(p,q) \ne (p_c, p_c/2)$, relies on the following preliminary result:

Lemma 5.1. *Assume that $p \in (p_c, 2)$, $q \in (p-1, q_\star)$, and consider u_0 satisfying* (1.8) *and*

$$0 \le u_0(x) \le K_0\ |x|^{-(p-q)/(q-p+1)}\,, \quad x \in \mathbb{R}^N\,, \tag{5.1}$$

for some $K_0 > 0$. Then, for $s \ge 0$ and $t > s$, we have

$$\|u(t)\|_1 \le C(u_0)\ \|u(t)\|_\infty^\theta\,, \tag{5.2}$$

with

$$\theta := (N+1)(q_\star - q)/(p-q)\,. \tag{5.3}$$

Assume further that $q \in (q_1, q_\star)$. Then

$$\|u(t)\|_1 \le C(u_0)\ \|u(s)\|_1^{q\xi\theta}\ (t-s)^{-N\xi\theta}\,, \tag{5.4}$$

where ξ is defined in (1.7)*.*

Proof. For $x \in \mathbb{R}^N$, $x \ne 0$, we define

$$\Sigma_{p,q}(x) := |x|^{-(p-q)/(q-p+1)} \quad \text{and} \quad A_0 := \frac{q-p+1}{p-q}\ \left(\frac{N(p-1)-q(N-1)}{q-p+1}\right)^{1/(q-p+1)}\,.$$

An easy computation shows that, for any $A \geq A_0$, $A\ \Sigma_{p,q}$ is a classical (stationary) supersolution to (1.1) in $\mathbb{R}^N \setminus \{0\}$. Owing to (5.1) $u_0 \leq A\ \Sigma_{p,q}$ for $A = \max\{K_0, A_0\}$ and the comparison principle ensures that

$$u(t,x) \leq A\ \Sigma_{p,q}(x)\,, \qquad (t,x) \in Q_\infty\,. \tag{5.5}$$

Since $q < q_\star$, it follows from (5.5) that, for $t > 0$ and $R > 0$, we have

$$\begin{aligned}
\|u(t)\|_1 &\leq \int_{B_R(0)} u(t,x)\ dx + \int_{\mathbb{R}^N \setminus B_R(0)} u(t,x)\ dx \\
&\leq C\ R^N\ \|u(t)\|_\infty + C(u_0) \int_R^\infty r^{N-1-((p-q)/(q-p+1))}\ dr \\
&\leq C(u_0)\ \left(R^N\ \|u(t)\|_\infty + R^{-(N+1)(q_\star - q)/(q-p+1)}\right)\,.
\end{aligned}$$

Choosing $R = (\|u(t)\|_\infty + \delta)^{-(q-p+1)/(p-q)}$ for $\delta \in (0,1)$, we obtain that

$$\|u(t)\|_1 \leq C\ \left(\|u(t)\|_\infty + \delta\right)^\theta\,,$$

the parameter θ being defined in (5.3). Since $\theta > 0$ and the above inequality is valid for all $\delta \in (0,1)$, we end up with (5.2) after letting $\delta \to 0$. We next combine (3.3) and (5.2) to deduce (5.4). $\square$

5.1 Improved decay

In this subsection we prove the first part of Theorem 1.2.

Proof of Theorem 1.2 (i): $p \in (p_c, 2)$ and $q \in (p/2, q_\star)$. Consider $T > 0$ and define

$$m(T) := \sup_{t \in (0,T]} \left\{ t^{(p-q)\theta/(2q-p)}\ \|u(t)\|_1 \right\}\,,$$

the parameter θ being defined in (5.3). Let $t \in (0,T]$. Since u_0 satisfies (1.13) and $q \in (q_1, q_\star)$, we infer from (5.4) with $s = t/2$ that

$$\begin{aligned}
t^{(p-q)\theta/(2q-p)}\ \|u(t)\|_1 &\leq C(u_0)\ \left\| u\left(\frac{t}{2}\right)\right\|_1^{q\xi\theta}\ t^{(p-q-N\xi(2q-p))\theta/(2q-p)} \\
&\leq C(u_0)\ \left\| u\left(\frac{t}{2}\right)\right\|_1^{q\xi\theta}\ \left(\frac{t}{2}\right)^{q(p-q)\xi\theta^2/(2q-p)} \\
&= C(u_0)\ \left\{ \left(\frac{t}{2}\right)^{(p-q)\theta/(2q-p)}\ \left\| u\left(\frac{t}{2}\right)\right\|_1 \right\}^{q\xi\theta} \\
&\leq C(u_0)\ m(T)^{q\xi\theta}\,.
\end{aligned}$$

The above estimate being valid for all $t \in (0,T]$, we conclude that $m(T) \leq C(u_0)\ m(T)^{q\xi\theta}$, whence $m(T) \leq C(u_0)$ since

$$q\xi\theta = 1 - \frac{N\xi(2q-p)}{p-q} < 1\,.$$

Since the constant $C(u_0)$ in the bound on $m(T)$ does not depend on $T > 0$, we have thus shown that

$$\|u(t)\|_1 \le C(u_0)\ t^{-(p-q)\theta/(2q-p)}\,, \qquad t > 0\,. \tag{5.6}$$

Combining (3.3) (with $s = t/2$) and (5.6) gives

$$\|u(t)\|_\infty \le C(u_0)\ t^{-(p-q)/(2q-p)}\,, \qquad t > 0\,,$$

and completes the proof of (1.14). □

5.2 Exponential decay

In this subsection we prove the second part of Theorem 1.2, which illustrates the role of branching point that our new (and initially unexpected) critical exponent $q = p/2$ plays on the large time behavior of solutions to (1.1).

Proof of Theorem 1.2 (ii): $p \in (p_c, 2)$ and $q = p/2$. In that case, the parameter θ defined in (5.3) satisfies $q\xi\theta = 1$, $N\xi\theta = 2N/p$, and, since $q \in (q_1, q_\star)$ and u_0 satisfies (1.13), it follows from (5.4) that

$$\|u(t)\|_1 \le C(u_0)\ (t-s)^{-2N/p}\ \|u(s)\|_1\,, \qquad 0 \le s < t\,. \tag{5.7}$$

Let $B > 0$ be a positive real number to be determined later, $T > B$ and define

$$m(T) := \sup_{t\in(0,T]} \left\{ e^{t/B}\ \|u(t)\|_1 \right\}.$$

If $t \in (B, T]$, we infer from (5.7) with $s = t - B \in (0, T]$ that

$$e^{t/B}\ \|u(t)\|_1 \le C(u_0)\ B^{-2N/p}\ e^{t/B}\ \|u(t-B)\|_1 \le C(u_0) e\ B^{-2N/p}\ m(T)\,,$$

while, if $t \in (0, B]$, we have $e^{t/B}\ \|u(t)\|_1 \le e\ \|u_0\|_1$. Therefore,

$$\begin{aligned} & e^{t/B}\ \|u(t)\|_1 \le e\ \|u_0\|_1 + C(u_0)\ B^{-2N/p}\ m(T)\,, \qquad t \in (0, T]\,, \\ & \left(1 - \frac{C(u_0)}{B^{2N/p}}\right)\ m(T) \le e\ \|u_0\|_1\,. \end{aligned}$$

Choosing B suitably large such that $B^{2N/p} \ge 2C(u_0)$ ensures that $m(T)$ is bounded from above by a positive constant which does not depend on T. Consequently, $\|u(t)\|_1 \le C(u_0)\ e^{-t/B}$ for $t \ge 0$ which implies together with (3.3) that $\|u(t)\|_\infty$ also decays at an exponential rate with a possibly different constant. □

We now show that, at least for $p > p_c$, the exponential decay obtained so far is optimal in the sense that the L^1-norm of u cannot decay faster than exponentially. More precisely, we have the following result:

Proposition 5.2. *If $p \in (p_c, 2)$, $q = p/2$, and u_0 satisfies (1.8), then there are positive constants $C_1(u_0)$ and $C_1'(u_0)$ depending on p, q, N, and u_0 such that*

$$\|u(t)\|_1 + \|u(t)\|_\infty \ge C_1'(u_0)\ e^{-C_1(u_0)t}\,, \quad t > 0\,. \tag{5.8}$$

In addition, $\mathcal{P} = Q_\infty$.

Proof. Let $t > 0$. By Proposition 4.1, we have

$$\frac{d}{dt}\|u(t)\|_1 + \int |\nabla u(t,x)|^{p/2}\, dx = 0\,,$$

while the gradient estimate (1.20) implies that

$$|\nabla u(t,x)| = \frac{p}{2-p}\, u^{2/p}(t,x)\, \left|\nabla u^{-(2-p)/p}(t,x)\right| \le C(u_0)\, u^{2/p}(t,x)\, \left(1 + t^{-1/p}\right)\,.$$

Combining the above two properties leads us to

$$0 \le \frac{d}{dt}\|u(t)\|_1 + C(u_0)\, \left(1 + t^{-1/p}\right)\, \|u(t)\|_1\,,$$

from which we readily conclude that $\|u(t)\|_1 \ge \|u_0\|_1\, e^{-C(u_0)(t+t^{1/p})}$ for $t \ge 0$. On the one hand, this implies that $\|u(t)\|_1 \ge \|u_0\|_1\, e^{-C(u_0)t}$ for $t \ge 1$, whence (5.8). On the other hand, we have $\|u(t)\|_1 > 0$ for all $t > 0$ and we proceed as in the proof of Corollary 4.5 to show that $u(t,x) > 0$ in Q_∞. $\square$

Proof of Proposition 1.8. We check the first assertion which readily follows from Proposition 4.4 and Corollary 4.5 when $p > p_c$ and $q > p/2$ and from Proposition 5.2 for $p > p_c$ and $q = p/2$. Consider next the case $p = p_c$ and $q > p_c/2$. A classical truncation argument ensures that there exists a non-negative compactly supported function $\tilde{u}_0$ satisfying (1.8) and $\tilde{u}_0 \le u_0$ in $\mathbb{R}^N$. Denoting the solution to (1.1) with initial condition $\tilde{u}_0$ by $\tilde{u}$, we infer from the comparison principle that $\tilde{u} \le u$ in Q_∞. In addition, $\tilde{u}_0$ obviously satisfies (4.9) for some $C_0 > 0$ and we are in a position to apply Proposition 4.4 and Corollary 4.5 to $\tilde{u}$ and deduce that $\|\tilde{u}(t)\|_1 > 0$ for all $t \ge 0$ and $\tilde{u} > 0$ in Q_∞. Consequently, u enjoys the same properties which completes the proof of the first assertion in Proposition 1.8.

Next, the second assertion follows from Proposition 4.7 if $q \in (0, q_\star]$ and from Proposition 4.6 if $p > p_c$ ad $q > q_\star$. Finally, if $p = p_c$ and $q > q_\star$, there is a non-negative compactly supported function $\tilde{u}_0$ satisfying (1.8) and $\tilde{u}_0 \le u_0$ in $\mathbb{R}^N$. On the one hand, the comparison principle guarantees that the solution $\tilde{u}$ to (1.1) with initial condition $\tilde{u}_0$ satisfies $\tilde{u} \le u$ in Q_∞. On the other hand, $\tilde{u}_0$ clearly satisfies (4.9) for a suitable constant C_0 and Proposition 4.6 ensures that $\lim_{t\to\infty} \|\tilde{u}(t)\|_1 > 0$. Combining these two facts completes the proof of Proposition 1.8. $\square$

5.3 Extinction

To complete the proof of Theorem 1.2, it remains to establish that finite time extinction takes place when $p \ge p_c$ and $q \in (0, p/2)$. To this end, we need to handle separately and by different methods the two cases: (a) $p \in (p_c, 2)$ and $q \in (q_1, p/2)$, (b) $p \in (p_c, 2)$ and $q \in (0, q_1]$. Let us begin with the case (a) for which the proof uses Lemma 5.1.

Proof of Theorem 1.2 (iii): $p \in (p_c, 2)$ and $q \in (q_1, p/2)$. In that case, we first observe that

$$N\xi\theta > q\xi\theta = 1 + \frac{N\xi(p-2q)}{p-q} > 1\,,$$

the parameter θ being still defined in (5.3). Setting $\lambda := q/(N\xi\theta(p-q))$ and recalling that $q < p-q$ as $q < p/2$ and u_0 satisfies (1.16) with $Q = q$, it follows from (5.4) that, for $s > 0$,

$$\begin{aligned} \tau(s) &:= \int_s^\infty \frac{\|u(t)\|_1^\lambda}{t}\ dt \le C(u_0)\ \|u(s)\|_1^{q\xi\theta\lambda} \int_s^\infty \frac{dt}{t(t-s)^{q/(p-q)}} \\ &\le C(u_0)\ (-\tau'(s))^{q\xi\theta}\ s^{q\xi(N+1)(p-2q)/(p-q)}\,, \end{aligned}$$

thus

$$\tau(s)^{1/(q\xi\theta)} \le -C(u_0)\ \tau'(s)\ s^{(p-2q)/(q_\star-q)}\,,$$

whence

$$\tau'(s) + C(u_0)\ s^{-(p-2q)/(q_\star-q)}\ \tau(s)^{1/(q\xi\theta)} \le 0\,, \qquad s > 0\,.$$

Since

$$\frac{p-2q}{q_\star-q} = 1 - \frac{1}{(N+1)\xi(q_\star-q)} < 1\,,$$

we infer from the above differential inequality that the function $\tilde{\tau} : s \mapsto \tau\left(s^{(N+1)\xi(q_\star-q)}\right)$ satisfies

$$\tilde{\tau}'(s) + C(u_0)\ \tilde{\tau}(s)^{1/(q\xi\theta)} \le 0\,, \qquad s > 0\,.$$

Since $q\xi\theta > 1$, we readily deduce from the above differential inequality that $\tilde{\tau}(s)$ vanishes identically for s large enough and so do $\tau(s)$ and $\|u(s)\|_1$. □

We next turn to the remaining case for $p > p_c$ for which we cannot use Lemma 5.1. We instead argue by comparison.

Proof of Theorem 1.2 (iii): $p \in (p_c, 2)$ and $q \in (0, q_1]$. In that case, $q_1 < p/2$ and, recalling that $Q \in (q_1, p/2)$ is defined in (1.16), we put

$$\tilde{u}(t,x) := \|\nabla u_0\|_\infty^{-(Q-q)/(Q-p+1)}\ u\left(\|\nabla u_0\|_\infty^{((2-p)(Q-q))/(Q-p+1)}\ t, x\right)\,, \qquad (t,x) \in Q_\infty\,.$$

It follows from (1.1), (1.2), and (2.8) that

$$\begin{aligned} \partial_t\tilde{u}(t,x) &= \|\nabla u_0\|_\infty^{-((p-1)(Q-q))/(Q-p+1)}\ \partial_t u\left(\|\nabla u_0\|_\infty^{((2-p)(Q-q))/(Q-p+1)}\ t, x\right) \\ &= \|\nabla u_0\|_\infty^{-((p-1)(Q-q))/(Q-p+1)}\ (\Delta_p u - |\nabla u|^q)\left(\|\nabla u_0\|_\infty^{((2-p)(Q-q))/(Q-p+1)}\ t, x\right) \\ &\le \Delta_p\tilde{u}(t,x) - \|\nabla u_0\|_\infty^{Q-q}\ |\nabla\tilde{u}(t,x)|^Q\ \left\|\nabla u\left(\|\nabla u_0\|_\infty^{((2-p)(Q-q))/(Q-p+1)}\ t\right)\right\|_\infty^{q-Q} \\ &\le \Delta_p\tilde{u}(t,x) - |\nabla\tilde{u}(t,x)|^Q\,, \end{aligned}$$

with $\tilde{u}(0) = U_0 := \|\nabla u_0\|_\infty^{-(Q-q)/(Q-p+1)}\ u_0$. Denoting the solution to (1.1)-(1.2) with Q instead of q and U_0 instead of u_0 by U, the comparison principle entails that $\tilde{u} \le U$ in Q_∞. As $Q \in (q_1, p/2)$ and u_0 satisfies (1.16), we already know that U has the finite time extinction property by Theorem 1.2. Consequently, $\tilde{u}$ and also u are identically zero after a finite time. □

The other two extinction ranges, either $p = p_c$ and $q \in (0, p_c/2)$, or $p \in (1, p_c)$ and $q > 0$, have been already considered in Theorem 1.1 and proved in Section 3.

5.4 A lower bound at the extinction time: $p \in (p_c, 2)$ and $q \in (q_1, p/2)$

It turns out that a simple modification of the proof of Theorem 1.2 for $p \in (p_c, 2)$ and $q \in (q_1, p/2)$ provides a lower bound on the L^1-norm and the L^∞-norm of $u(t)$ as t approaches the extinction time T_{e}.

Proposition 5.3. *Assume that $p \in (p_c, 2)$, $q \in (q_1, p/2)$, and that u_0 satisfies* (1.8) *and* (1.16) *(with $Q = q$). Denoting the extinction time of the corresponding solution u to* (1.1)-(1.2) *by T_e, we have*

$$\begin{aligned} C\ (T_e - t)^{(N+1)(q_\star - q)/(p-2q)} &\le \|u(t)\|_1\,, \qquad t \in (0, T_e)\,, &(5.9)\\ C\ (T_e - t)^{(p-q)/(p-2q)} &\le \|u(t)\|_\infty\,, \qquad t \in (0, T_e)\,. &(5.10)\end{aligned}$$

Proof. By Theorem 1.2 (iii), T_{e} is finite and $\|u(t)\|_1 > 0$ for $t \in [0, T_{\mathrm{e}})$. Setting $\lambda = q/(N\xi\theta(p-q))$ with θ defined in (5.3) and recalling that $q < (p-q)$ as $q < p/2$, it follows from (5.4) that, for $s \in (0, T_{\mathrm{e}})$,

$$\begin{aligned} \tau(s) &:= \int_s^{T_{\mathrm{e}}} \|u(t)\|_1^\lambda\ dt \le C(u_0)\ \|u(s)\|_1^{q\xi\theta\lambda} \int_s^{T_{\mathrm{e}}} \frac{dt}{(t-s)^{q/(p-q)}} \\ &\le C(u_0)\ (-\tau'(s))^{q\xi\theta}\ (T_{\mathrm{e}} - s)^{(p-2q)/(p-q)}\,, \end{aligned}$$

from which we deduce the following differential inequality:

$$\tau(s)^{1/(q\xi\theta)} \le -C(u_0)\ \tau'(s)\ (T_{\mathrm{e}} - s)^{(p-2q)/(q(N+1)\xi(q_\star - q))}\,,$$

whence

$$\tau'(s) + C(u_0)\ (T_{\mathrm{e}} - s)^{-(p-2q)/(q(N+1)\xi(q_\star - q))}\ \tau(s)^{1/(q\xi\theta)} \le 0\,, \qquad s \in (0, T_{\mathrm{e}})\,.$$

Since

$$\frac{1}{q\xi\theta} = 1 - \frac{N(p-2q)}{q(N+1)(q_\star - q)} < 1 \quad \text{and} \quad \frac{p-2q}{q(N+1)\xi(q_\star - q)} = 1 - \frac{N\xi(p-2q) + q}{q(N+1)\xi(q_\star - q)} < 1\,,$$

the above differential inequality also reads

$$\frac{d}{ds}\left[\tau(s)^{N(p-2q)/(q(N+1)(q_\star - q))} - C(u_0)\ (T_{\mathrm{e}} - s)^{(N\xi(p-2q)+q)/(q(N+1)\xi(q_\star - q))}\right] \le 0$$

for $s \in (0, T_{\mathrm{e}})$. Integrating the above inequality with respect to s over (t, T_{e}) for $t \in (0, T_{\mathrm{e}})$ gives

$$\begin{aligned} C(u_0)\ (T_{\mathrm{e}} - t)^{(N\xi(p-2q)+q)/(q(N+1)\xi(q_\star - q))} &\le \tau(t)^{N(p-2q)/(q(N+1)(q_\star - q))}\,, \\ C(u_0)\ (T_{\mathrm{e}} - t)^{(N\xi(p-2q)+q)/(N\xi(p-2q))} &\le \tau(t)\,. &(5.11)\end{aligned}$$

Owing to the time monotonicity (6.3) of $\|u\|_1$, we have

$$\tau(t) = \int_t^{T_{\mathrm{e}}} \|u(s)\|_1^\lambda\ ds \le (T_{\mathrm{e}} - t)\ \|u(t)\|_1^\lambda\,, \qquad t \in (0, T_{\mathrm{e}})\,. \tag{5.12}$$

Combining (5.11) and (5.12) gives (5.9). Next, (5.10) readily follows from (5.2) and (5.9). ∎

6 Well-posedness

In this section we study the existence and uniqueness of a solution to (1.1)-(1.2). This is done through an approximation process, in order to avoid the singularity in the diffusion.

We begin by stating in a precise form the notion of a viscosity solution to the singular equation (1.1). The standard definition has been adapted to deal with singular equations in [19, 24], by restricting the comparison functions. We follow their approach. Let $\mathcal{F}$ be the set of functions $f \in C^2([0,\infty))$ satisfying

$$f(0) = f'(0) = f''(0) = 0, \;\; f''(r) > 0 \text{ for all } r > 0, \quad \lim_{r\to 0} |f'(r)|^{p-2} f''(r) = 0.$$

For example, $f(r) = r^\sigma$ with $\sigma > p/(p-1) > 2$ belongs to $\mathcal{F}$. We introduce then the class $\mathcal{A}$ of admissible comparison functions $\psi \in C^2(Q_\infty)$ defined as follows: $\psi \in \mathcal{A}$ if, for any $(t_0, x_0) \in Q_\infty$ where $\nabla\psi(t_0,x_0) = 0$, there exist a constant $\delta > 0$, a function $f \in \mathcal{F}$, and a modulus of continuity $\omega \in C([0,\infty))$, (that is, a non-negative function satisfying $\omega(r)/r \to 0$ as $r \to 0$), such that, for all $(t,x) \in Q_\infty$ with $|x - x_0| + |t - t_0| < \delta$, we have

$$|\psi(t,x) - \psi(t_0,x_0) - \partial_t\psi(t_0,x_0)(t-t_0)| \le f(|x-x_0|) + \omega(|t-t_0|).$$

Definition 6.1. *An upper semicontinuous function $u : Q_\infty \to \mathbb{R}$ is a viscosity subsolution to* (1.1) *in Q_∞ if, whenever $\psi \in \mathcal{A}$ and $(t_0,x_0) \in Q_\infty$ are such that*

$$u(t_0,x_0) = \psi(t_0,x_0), \quad u(t,x) < \psi(t,x), \; \textit{for all } (t,x) \in Q_\infty \setminus \{(t_0,x_0)\},$$

then

$$\begin{cases} \partial_t\psi(t_0,x_0) \le \Delta_p\psi(t_0,x_0) - |\nabla\psi(t_0,x_0)|^q & \textit{if } \nabla\psi(t_0,x_0) \ne 0, \\ \partial_t\psi(t_0,x_0) \le 0 & \textit{if } \nabla\psi(t_0,x_0) = 0. \end{cases} \tag{6.1}$$

A lower semicontinuous function $u : Q_\infty \to \mathbb{R}$ is a viscosity supersolution to (1.1) *in Q_∞ if $-u$ is a viscosity subsolution to* (1.1) *in Q_∞. A continuous function $u : Q_\infty \to \mathbb{R}$ is a viscosity solution to* (1.1) *in Q_∞ if it is a viscosity subsolution and supersolution.*

We refer to [24] for basic results about viscosity solutions; in particular the comparison principle is [24, Theorem 3.9] and the stability property with respect to uniform limits is [24, Theorem 6.1]. We are now ready to state the main result of this section.

Theorem 6.2. *Given an initial condition u_0 satisfying* (1.8) *there is a unique non-negative viscosity solution u to* (1.1)-(1.2) *which satisfies the gradient estimates stated in Theorems 1.3, 1.5 and 1.7 according to the range of (p,q). In addition, u is a weak solution to* (1.1)-(1.2)*, that is,*

$$\int (u(t,x) - u(s,x))\, \vartheta(x)\, dx + \int_s^t \int \left(|\nabla u|^{p-2}\, \nabla u \cdot \nabla\vartheta + |\nabla u|^q\, \vartheta \right)\, dx d\tau = 0 \tag{6.2}$$

for $t > s \ge 0$ and all $\vartheta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ and satisfies

$$\|u(t)\|_1 + \int_s^t \int |\nabla u(\tau,x)|^q\, dx d\tau \le \|u(s)\|_1\,. \tag{6.3}$$

Remark 6.3. *In fact the existence result can be extended to a larger class of initial data, namely $u_0 \in BC(\mathbb{R}^N)$. This can be proved by further regularization and arguing as in [14].*

The rest of the section is devoted to the proof of Theorem 6.2. This will be divided into several steps.

6.1 Approximation

In a first step, we have to introduce a regularization of (1.1) in order to avoid the problems coming from the singularity at points where $\nabla u = 0$ and from the possible lack of regularity of the solutions. For $\varepsilon \in (0,1/2)$, we let

$$a_\varepsilon(\xi) := (\xi+\varepsilon^2)^{(p-2)/2}, \quad b_\varepsilon(\xi) := (\xi+\varepsilon^2)^{q/2} - \varepsilon^q, \qquad \xi \geq 0\,, \tag{6.4}$$

and consider the following Cauchy problem

$$\left\{ \begin{array}{l} \partial_t u_\varepsilon - \mathrm{div}(a_\varepsilon(|\nabla u_\varepsilon|^2)\nabla u_\varepsilon) + b_\varepsilon(|\nabla u_\varepsilon|^2) = 0,\ (t,x) \in Q_\infty\,, \\ u_\varepsilon(0,x) = u_{0\varepsilon}(x) + \varepsilon^\gamma,\ x \in \mathbb{R}^N\,, \end{array} \right. \tag{6.5}$$

where $\gamma \in (0,p/4) \cap (0,q/2)$ is a small parameter such that $\gamma < \min\{p-1, 1-k\}$ and $u_{0\varepsilon} \in C^\infty(\mathbb{R}^N)$ is a non-negative smooth approximation of u_0 satisfying

$$\|u_{0\varepsilon}\|_\infty \leq \|u_0\|_\infty \quad \text{and} \quad \|\nabla u_{0\varepsilon}\|_\infty \leq (1 + C(u_0)\varepsilon)\|\nabla u_0\|_\infty \tag{6.6}$$

and such that $(u_{0\varepsilon})$ converges to u_0 uniformly in compact subsets of $\mathbb{R}^N$. Further smallness conditions on γ and ε will appear in the sequel and will be stated wherever needed. By standard existence results for quasilinear parabolic equations [22], (6.5) has a unique classical solution $u_\varepsilon \in C^{(3+\delta)/2, 3+\delta}([0,\infty) \times \mathbb{R}^N)$ for some $\delta \in (0,1)$. By comparison with constant solutions ε^γ and $\varepsilon^\gamma + \|u_0\|_\infty$, we find

$$\varepsilon^\gamma \leq u_\varepsilon(t,x) \leq \varepsilon^\gamma + \|u_0\|_\infty,\ (t,x) \in Q_\infty. \tag{6.7}$$

We now turn to estimates for the gradient of u_ε. Let φ be a C^3-smooth monotone function with inverse $\psi = \varphi^{-1}$ and set $\varrho = 1/\psi'$. Defining $v_\varepsilon := \varphi^{-1}(u_\varepsilon)$ and $w_\varepsilon := |\nabla v_\varepsilon|^2$, the regularity of a_ε, b_ε, and u_ε allows us to apply [2, Lemma 2.1] and obtain that w_ε satisfies the differential inequality

$$\partial_t w_\varepsilon - A_\varepsilon w_\varepsilon - B_\varepsilon \cdot \nabla w_\varepsilon + 2\tilde{R}_1^\varepsilon\ w_\varepsilon^2 + 2\tilde{R}_2^\varepsilon\ w_\varepsilon \leq 0 \ \text{ in } \ Q_\infty\,, \tag{6.8}$$

with

$$\begin{array}{rcl} A_\varepsilon w_\varepsilon & := & a_\varepsilon \Delta w_\varepsilon + 2a_\varepsilon'\ (\nabla u_\varepsilon)^t D^2 w_\varepsilon \nabla u_\varepsilon\,, \\ \tilde{R}_1^\varepsilon & := & -a_\varepsilon \left(\dfrac{\varphi''}{\varphi'} \right)' - \left((N-1)\ \dfrac{(a_\varepsilon')^2}{a_\varepsilon} + 4a_\varepsilon'' \right)\ (\varphi'\varphi'')^2\ w_\varepsilon^2 \\ & & -2\ a_\varepsilon'\ \left(2(\varphi'')^2 + \varphi'\ \varphi'''\right) w_\varepsilon\,, \\ \tilde{R}_2^\varepsilon & := & \dfrac{\varphi''}{(\varphi')^2}\ \left(2b_\varepsilon'(\varphi')^2 w_\varepsilon - b_\varepsilon\right)\,, \end{array}$$

in which we have omitted to write the dependence of a_ε and b_ε upon $|\nabla u_\varepsilon|^2$ and that of φ upon v_ε. Setting $g_\varepsilon := (|\nabla u_\varepsilon|^2 + \varepsilon^2)^{1/2}$, we have $|\nabla u_\varepsilon|^2 = g_\varepsilon^2 - \varepsilon^2$ and we proceed as in Section 2 to compute $\tilde{R}_1^\varepsilon$ and $\tilde{R}_2^\varepsilon$:

$$\begin{array}{l} \tilde{R}_1^\varepsilon := (p-1)\ R_1^\varepsilon + \varepsilon^2\ R_{11}^\varepsilon \ \text{ with } \ R_1^\varepsilon := g_\varepsilon^{p-2} \left[k\ \varrho'(u_\varepsilon)^2 - (\varrho\varrho'')(u_\varepsilon) \right]\,, \\ \tilde{R}_2^\varepsilon := (q-1)\ R_2^\varepsilon + R_{21}^\varepsilon \ \text{ with } \ R_2^\varepsilon := \left(\dfrac{\varrho'}{\varrho} \right) (u_\varepsilon)\ g_\varepsilon^q, \end{array} \tag{6.9}$$

and

$$
\begin{aligned}
R_{11}^\varepsilon &:= \left[(p-2)\varrho\varrho'' - (p-1)k(\varrho')^2\right](u_\varepsilon)\ g_\varepsilon^{p-4} \\
&\quad + \frac{(2-p)[2(N+7)-p(N+3)]}{4}\ \varrho'(u_\varepsilon)^2\ g_\varepsilon^{p-6}\ (g_\varepsilon^2-\varepsilon^2), \\
R_{21}^\varepsilon &:= \left(\frac{\varrho'}{\varrho}\right)(u_\varepsilon)\ \left(\varepsilon^q - q\varepsilon^2\ g_\varepsilon^{q-2}\right),
\end{aligned}
\tag{6.10}
$$

After these preliminary calculations, we are ready to prove gradient estimates for u_ε, that will give a rigorous proof of the gradient estimates listed in Theorems 1.3, 1.5, and 1.7 after passing to the limit $\varepsilon \to 0$ and a tool in the proof of well-posedness. Before the more sophisticated estimates, let us notice that, taking $\varrho(r) \equiv 1$, we have $R_1^\varepsilon = R_{11}^\varepsilon = R_2^\varepsilon = R_{21}^\varepsilon = 0$ and the comparison principle applied to (6.8) and combined with (6.6) readily gives

$$
\|\nabla u_\varepsilon(t)\|_\infty \le \|\nabla u_{0\varepsilon}\|_\infty \le (1+C(u_0)\varepsilon)\|\nabla u_0\|_\infty\,, \quad t\ge 0. \tag{6.11}
$$

Consequently,

$$
\varepsilon \le g_\varepsilon \le \varepsilon + \|\nabla u_{0\varepsilon}\|_\infty \le \|\nabla u_0\|_\infty + C(u_0)\varepsilon \quad \text{in} \quad Q_\infty. \tag{6.12}
$$

6.2 Gradient estimates

In this subsection, we prove gradient estimates for u_ε. We divide the proof into the same cases as in Section 2. In all cases, we will follow the four-step scheme: first estimate the extra term $R_{11}^\varepsilon w_\varepsilon$, then the influence of the diffusion term $R_1^\varepsilon w_\varepsilon^2$, then (if needed) the influence of the absorption terms $R_2^\varepsilon w_\varepsilon$ and $R_{21}^\varepsilon w_\varepsilon$ and finally find a suitable supersolution, as in the formal derivation performed in Section 2.

6.2.1 $p > p_c$ and $q \ge p/2$.

As in Section 2.1 we choose

$$
\varrho(z) = \left(\frac{p^2}{2(2k+p-2)}\right)^{1/p}\ z^{2/p},
$$

and we obtain

$$
\begin{aligned}
R_{11}^\varepsilon = \left(\frac{p^2}{2(2k+p-2)}\right)^{2/p} u_\varepsilon^{(4-2p)/p}\ g_\varepsilon^{p-4} \Bigg\{ &-\frac{2(2-p)^2}{p^2} - \frac{4k(p-1)}{p^2} \\
&+\frac{(2-p)[2(N+7)-p(N+3)]}{p^2}\ \frac{g_\varepsilon^2-\varepsilon^2}{g_\varepsilon^2}\Bigg\} \ge -C\ u_\varepsilon^{(4-2p)/p}\ g_\varepsilon^{p-4},
\end{aligned}
$$

hence, since

$$
w_\varepsilon = \left|\nabla\varphi^{-1}(u_\varepsilon)\right|^2 = \frac{|\nabla u_\varepsilon|^2}{\varrho(u_\varepsilon)^2} = \frac{g_\varepsilon^2-\varepsilon^2}{\varrho(u_\varepsilon)^2}\,, \tag{6.13}
$$

$$
R_{11}^\varepsilon w_\varepsilon \ge -C\ u_\varepsilon^{(4-2p)/p}\ g_\varepsilon^{p-4}\ \frac{g_\varepsilon^2-\varepsilon^2}{\varrho(u_\varepsilon)^2} = -C\ g_\varepsilon^{p-4}(g_\varepsilon^2-\varepsilon^2)\ u_\varepsilon^{-2} \ge -\frac{C}{\varepsilon^{2\gamma}}\ g_\varepsilon^{p-2}
$$

by (6.7). Thus, from the formula (6.9), we deduce

$$\begin{aligned}
\tilde{R}_1^\varepsilon\ w_\varepsilon^2 &\geq (p-1)\left(\frac{p^2}{2(2k+p-2)}\right)^{(2-p)/p} g_\varepsilon^{p-2} u_\varepsilon^{(4-2p)/p} w_\varepsilon^2 - C_1 \varepsilon^{2(1-\gamma)} g_\varepsilon^{p-2} w_\varepsilon \\
&\geq (p-1)\left(\frac{p^2}{2(2k+p-2)}\right)^{(2-p)/p} u_\varepsilon^{(4-2p)/p} \frac{g_\varepsilon^p - \varepsilon^2 g_\varepsilon^{p-2}}{\varrho(u_\varepsilon)^2} w_\varepsilon - C_1 \varepsilon^{p-2\gamma} w_\varepsilon \\
&\geq (p-1)\left(\frac{p^2}{2(2k+p-2)}\right)^{(2-p)/p} u_\varepsilon^{(4-2p)/p} \frac{g_\varepsilon^p - \varepsilon^p}{\varrho(u_\varepsilon)^2} w_\varepsilon - C_1 \varepsilon^{p-2\gamma} w_\varepsilon \\
&\geq (p-1)\left(\frac{p^2}{2(2k+p-2)}\right)^{(2-p)/p} u_\varepsilon^{(4-2p)/p} \varrho(u_\varepsilon)^{p-2} w_\varepsilon^{1+p/2} - C_1 u_\varepsilon^{(4-2p)/p} \frac{\varepsilon^p}{\varrho(u_\varepsilon)^2} w_\varepsilon \\
&\quad - C_1 \varepsilon^{p-2\gamma} w_\varepsilon \geq (p-1)\ w_\varepsilon^{1+p/2} - C_1 \varepsilon^{p-2\gamma} w_\varepsilon,
\end{aligned}$$

where we have repeatedly used the lower bound in (6.12), (6.7), and (6.13). We also have

$$\tilde{R}_2^\varepsilon w_\varepsilon = C u_\varepsilon^{-1} \left[\varepsilon^q - q\varepsilon^2 g_\varepsilon^{q-2} - (1-q) g_\varepsilon^q\right] w_\varepsilon. \tag{6.14}$$

We need to treat in a different way the cases $q > 1$ and $q < 1$.

If $q > 1$, we notice that $\tilde{R}_2^\varepsilon w_\varepsilon \geq 0$. Indeed, for $q \geq 1$, we have $q\varepsilon^2 g_\varepsilon^{q-2} \leq q\varepsilon g_\varepsilon^{q-1} \leq \varepsilon^q + (q-1) g_\varepsilon^q$ by Young's inequality. Hence, we can simply drop the effect of this term and deduce from (6.8) and the previous lower bound on $\tilde{R}_1^\varepsilon$ that

$$L_\varepsilon w_\varepsilon := \partial_t w_\varepsilon - A_\varepsilon w_\varepsilon - B_\varepsilon \cdot \nabla w_\varepsilon + 2(p-1)\ w_\varepsilon^{1+p/2} - C_1 \varepsilon^{p-2\gamma} w_\varepsilon \leq 0$$

in Q_∞. It is then straightforward to check that the function

$$W_\varepsilon(t) = \left(\frac{2 + pC_1 \varepsilon^{p/2}}{2p(p-1)}\right)^{2/p} t^{-2/p}$$

is a supersolution for the differential operator L_ε above in $(0, \varepsilon^{(4\gamma-p)/2}) \times \mathbb{R}^N$, provided we choose $\gamma < p/4$. The comparison principle and the definition (1.7) of k then ensure that

$$\left|\nabla u_\varepsilon^{-(2-p)/p}(t,x)\right| \leq \left(\frac{2-p}{p}\right)^{(p-1)/p} \eta^{1/p} \left(1 + C_1 \varepsilon^{p/2}\right)^{1/p} t^{-1/p} \tag{6.15}$$

for any $(t,x) \in (0, \varepsilon^{(4\gamma-p)/2}) \times \mathbb{R}^N$. Notice that $4\gamma - p < 0$ by the choice of γ, so that the time interval of validity of (6.15) increases to $(0,\infty)$ as $\varepsilon \to 0$.

If $q \in [p/2, 1)$, we can further estimate the right-hand side of (6.14), taking into account the lower bound $g_\varepsilon > \varepsilon$, which implies

$$R_{21}^\varepsilon \geq C\ u_\varepsilon^{-1}\ \left(\varepsilon^q - q\varepsilon^2 g_\varepsilon^{q-2}\right) \geq (1-q) C\ u_\varepsilon^{-1}\ \varepsilon^q \geq 0,$$

while (6.7) and (6.13) give

$$\begin{aligned}
(q-1)\ R_2^\varepsilon w_\varepsilon &\geq -C_3\ u_\varepsilon^{-1}\ g_\varepsilon^q\ w_\varepsilon \geq -C_3\ u_\varepsilon^{-1}\ \left(\varepsilon^2 + \varrho(u_\varepsilon)^2 w_\varepsilon\right)^{q/2}\ w_\varepsilon \\
&\geq -C_3\ u_\varepsilon^{-1}\ \left(\varepsilon^q w_\varepsilon + \varrho(u_\varepsilon)^q w_\varepsilon^{(2+q)/2}\right) \\
&\geq -C_3\ \varepsilon^{q-\gamma} w_\varepsilon - C_3\ \left(\|u_0\|_\infty + \varepsilon^\gamma\right)^{(2q-p)/p}\ w_\varepsilon^{(2+q)/2},
\end{aligned}$$

where we have used the form of ϱ and (6.7). Combining this lower bound with the already obtained lower bound on $\tilde{R}_1^\varepsilon$, we obtain

$$
\begin{aligned}
L_\varepsilon w_\varepsilon := \partial_t w_\varepsilon - A_\varepsilon w_\varepsilon - B_\varepsilon \cdot \nabla w_\varepsilon + C_1 w_\varepsilon^{(2+p)/2} &- C_3(\|u_0\|_\infty + \varepsilon^\gamma)^{(2q-p)/p} w_\varepsilon^{(2+q)/2} \\
&- C_4 \left(\varepsilon^{q-\gamma} + \varepsilon^{p-2\gamma}\right) w_\varepsilon \le 0
\end{aligned}
$$

in Q_∞. We notice that the function

$$
W_\varepsilon(t) = C_5 \left[(\|u_0\|_\infty + \varepsilon^\gamma)^{2(2q-p)/p(p-q)} + \varepsilon^{2(p-2\gamma)/p} + \varepsilon^{2(q-\gamma)/p} \right] + \left(\frac{4}{pC_1} \right)^{2/p} t^{-2/p}
$$

is a supersolution for the differential operator L_ε in Q_∞ for a sufficiently large constant C_5. By the comparison principle, we obtain the following gradient estimate:

$$
\left| \nabla u_\varepsilon^{-(2-p)/p}(t,x) \right| \le C \left[(\|u_0\|_\infty + \varepsilon^\gamma)^{(2q-p)/p(p-q)} + \varepsilon^{(p-2\gamma)/p} + \varepsilon^{(q-\gamma)/p} + t^{-1/p} \right] \tag{6.16}
$$

for any $(t,x) \in Q_\infty$.

6.2.2 $p > p_c$ and $q \in (0, p/2)$.

As in Section 2.2, we choose the following function

$$
\varrho(z) = \left(\frac{p-q}{k+p-q-1} \right)^{1/(p-q)} z^{1/(p-q)},
$$

recalling that $k+p-q-1 > 0$ in that case. We estimate $\tilde{R}_1^\varepsilon$ and $\tilde{R}_2^\varepsilon$ in the same way as in Section 6.2.1, the only significant difference stemming from the special form of ϱ. We have

$$
\begin{aligned}
R_{11}^\varepsilon = \left(\frac{p-q}{k+p-q-1} \right)^{2/(p-q)} & \frac{1}{(p-q)^2} g_\varepsilon^{p-4} u_\varepsilon^{2(q-p+1)/(p-q)} \Big[-(2-p)(q-p+1) - k(p-1) \\
& + \frac{(2-p)(2(N+7) - p(N+3))}{4} \frac{g_\varepsilon^2 - \varepsilon^2}{g_\varepsilon^2} \Big] \ge -C u_\varepsilon^{2(q-p+1)/(p-q)} g_\varepsilon^{p-4},
\end{aligned}
$$

hence $R_{11}^\varepsilon w_\varepsilon \ge -C_1 \varepsilon^{-2\gamma} g_\varepsilon^{p-2}$, a similar estimate as in Section 6.2.1 (and with exactly the same proof relying on (6.7) and (6.13)). Consequently, following the same steps as in Section 6.2.1,

$$
\begin{aligned}
\tilde{R}_1^\varepsilon\ w_\varepsilon^2 &\ge C_2 g_\varepsilon^{p-2} u_\varepsilon^{2(q-p+1)/(p-q)} w_\varepsilon^2 - C_1 \varepsilon^{2(1-\gamma)} g_\varepsilon^{p-2} w_\varepsilon \\
&\ge C_2 u_\varepsilon^{2(q-p+1)/(p-q)} \frac{g_\varepsilon^p - \varepsilon^2 g_\varepsilon^{p-2}}{\varrho(u_\varepsilon)^2} w_\varepsilon - C_1 \varepsilon^{p-2\gamma} w_\varepsilon \\
&\ge C_2 u_\varepsilon^{2(q-p+1)/(p-q)} \frac{g_\varepsilon^p - \varepsilon^p}{\varrho(u_\varepsilon)^2} w_\varepsilon - C_1 \varepsilon^{p-2\gamma} w_\varepsilon \\
&\ge C_2 u_\varepsilon^{2(q-p+1)/(p-q)} \varrho(u_\varepsilon)^{p-2} w_\varepsilon^{(2+p)/2} - C_2 u_\varepsilon^{2(q-p+1)/(p-q)} \frac{\varepsilon^p}{\varrho(u_\varepsilon)^2} w_\varepsilon - C_1 \varepsilon^{p-2\gamma} w_\varepsilon \\
&\ge C_2 u_\varepsilon^{(2q-p)/(p-q)} w_\varepsilon^{(2+p)/2} - C_1 \varepsilon^{p-2\gamma} w_\varepsilon.
\end{aligned}
$$

We next estimate $\tilde{R}_2^\varepsilon$:

$$\begin{aligned}\tilde{R}_2^\varepsilon w_\varepsilon &= C_3 u_\varepsilon^{-1}\left[\varepsilon^q - q\varepsilon^2 g_\varepsilon^{q-2} - (1-q) g_\varepsilon^q\right] w_\varepsilon \geq -C_3 u_\varepsilon^{-1} g_\varepsilon^q w_\varepsilon \\ &\geq -C_3 u_\varepsilon^{-1} (\varepsilon^2 + \varrho(u_\varepsilon)^2 w_\varepsilon)^{q/2} w_\varepsilon \geq -C_3 u_\varepsilon^{-1}\left(\varepsilon^q w_\varepsilon + \varrho(u_\varepsilon)^q w_\varepsilon^{(2+q)/2}\right) \\ &\geq -C_3\left[\varepsilon^{q-\gamma} w_\varepsilon + u_\varepsilon^{(2q-p)/(p-q)} w_\varepsilon^{(2+q)/2}\right].\end{aligned}$$

From (6.8) and these estimates, and taking into account that $\gamma < p/2 < p-q$, we obtain that

$$L_\varepsilon w_\varepsilon := \partial_t w_\varepsilon - A_\varepsilon w_\varepsilon - B_\varepsilon \cdot \nabla w_\varepsilon + C_2 u_\varepsilon^{(2q-p)/(p-q)} w_\varepsilon^{(2+q)/2}\left(w_\varepsilon^{(p-q)/2} - C_4\right) - C_5 \varepsilon^{q-\gamma} w_\varepsilon \leq 0.$$

We look for a supersolution for L_ε of the form $W_\varepsilon(t) = \lambda + \mu t^{-2/p}$. Proceeding as in Sections 2.2 and 6.2.1, we find that

$$\begin{aligned}W_\varepsilon(t) &= \left(\frac{4C_5}{C_2}\right)^{2/p} (\|u_0\|_\infty + \varepsilon^\gamma)^{2(p-2q)/p(p-q)}\ \varepsilon^{2(q-\gamma)/p} + (4C_4)^{2/(p-q)} \\ &+ \left(\frac{4}{pC_2}\right)^{2/p} (\|u_0\|_\infty + \varepsilon^\gamma)^{2(p-2q)/p(p-q)}\ t^{-2/p}\end{aligned}$$

is a supersolution in Q_∞. We thus obtain the following gradient estimate

$$|\nabla u_\varepsilon(t,x)| u_\varepsilon(t,x)^{-1/(p-q)} \leq C\left[1 + (\|u_0\|_\infty + \varepsilon^\gamma)^{(p-2q)/p(p-q)}\left(\varepsilon^{(q-\gamma)/p} + t^{-1/p}\right)\right], \quad (6.17)$$

for any $(t,x) \in Q_\infty$. This is the approximation of (2.15), and the discussion with respect to the sign of $p-1-q$ is the same as in Section 2.2 and is omitted here.

6.2.3 $p = p_c$.

We follow the same general strategy as in the previous cases. The computations are slightly different since logarithmic terms appear in the choice of ϱ.

For $q > p_c/2$, we take

$$\varrho(z) = z^{(N+1)/N} (\log M_\varepsilon - \log z)^{(N+1)/2N}, \quad M_\varepsilon = e(\|u_0\|_\infty + \varepsilon^\gamma).$$

Let us notice first that, by (6.7),

$$1 \leq \log M_\varepsilon - \log u_\varepsilon. \quad (6.18)$$

On the one hand, owing to (6.18),

$$\begin{aligned}R_1^\varepsilon &= \frac{N+1}{4N}\ u_\varepsilon^{2/N}\ \left[2(\log M_\varepsilon - \log u_\varepsilon)^{1/N} + (\log M_\varepsilon - \log u_\varepsilon)^{-(N-1)/N}\right] \\ &\geq \frac{N+1}{2N}\ u_\varepsilon^{2/N}\ g_\varepsilon^{p_c-2} (\log M_\varepsilon - \log u_\varepsilon)^{1/N}.\end{aligned}$$

On the other hand, after direct, but rather long computations, and dropping, as usual, the last term in the expression (6.10) of R_{11}^ε, we deduce from (6.18) that

$$\begin{aligned}R_{11}^\varepsilon &\geq \frac{N+1}{2N^2} u_\varepsilon^{2/N} g_\varepsilon^{p_c-4}\Bigg[-2(\log M_\varepsilon - \log u_\varepsilon)^{(N+1)/N} + \frac{4N+2}{N+1}\ (\log M_\varepsilon - \log u_\varepsilon)^{1/N} \\ &\quad + \frac{N-1}{2(N+1)} (\log M_\varepsilon - \log u_\varepsilon)^{-(N-1)/N}\Bigg] \\ &\geq -\frac{N+1}{N^2}\ u_\varepsilon^{2/N}\ g_\varepsilon^{p_c-4} (\log M_\varepsilon - \log u_\varepsilon)^{(N+1)/N},\end{aligned}$$

Consequently, thanks to (6.7) and (6.13), we have

$$\begin{aligned} R_{11}^\varepsilon w_\varepsilon &\geq -\frac{N+1}{N^2}\ u_\varepsilon^{2/N}\ g_\varepsilon^{p_c-4}\ (\log M_\varepsilon - \log u_\varepsilon)^{(N+1)/N}\ \frac{g_\varepsilon^2-\varepsilon^2}{\varrho(u_\varepsilon)^2} \\ &\geq -C_1 u_\varepsilon^{-2} g_\varepsilon^{p_c-4}(g_\varepsilon^2-\varepsilon^2) \geq -C_1\ \varepsilon^{-2\gamma} g_\varepsilon^{p_c-2}. \end{aligned}$$

Using again (6.7), (6.12), (6.13), and (6.18), we can now estimate

$$\begin{aligned} \tilde{R}_1^\varepsilon w_\varepsilon^2 &\geq C_2 g_\varepsilon^{p_c-2} u_\varepsilon^{2/N}\ (\log M_\varepsilon - \log u_\varepsilon)^{1/N}\ w_\varepsilon^2 - C_1 \varepsilon^{2(1-\gamma)} g_\varepsilon^{p_c-2} w_\varepsilon \\ &\geq C_2 u_\varepsilon^{2/N} \frac{g_\varepsilon^{p_c} - \varepsilon^2 g_\varepsilon^{p_c-2}}{\varrho(u_\varepsilon)^2}\ (\log M_\varepsilon - \log u_\varepsilon)^{1/N}\ w_\varepsilon - C_1 \varepsilon^{p_c-2\gamma} w_\varepsilon \\ &\geq C_2 u_\varepsilon^{2/N}\ (\log M_\varepsilon - \log u_\varepsilon)^{1/N}\ \left[\varrho(u_\varepsilon)^{p_c-2}\ w_\varepsilon^{(2+p_c)/2} - \frac{\varepsilon^{p_c}}{\varrho(u_\varepsilon)^2} w_\varepsilon\right] - C_1 \varepsilon^{p_c-2\gamma} w_\varepsilon \\ &\geq C_2 w_\varepsilon^{(2+p_c)/2} - C_2 u_\varepsilon^{-2}\ (\log M - \log u_\varepsilon)^{-1} \varepsilon^{p_c}\ w_\varepsilon - C_1 \varepsilon^{p_c-2\gamma}\ w_\varepsilon \\ &\geq C_2 w_\varepsilon^{(2+p_c)/2} - C_1 \varepsilon^{p_c-2\gamma} w_\varepsilon. \end{aligned}$$

It remains to estimate $\tilde{R}_2^\varepsilon$. By direct computation, we find

$$\tilde{R}_2^\varepsilon = \frac{N+1}{2Nu_\varepsilon}\ \frac{2\log M_\varepsilon - 2\log u_\varepsilon - 1}{\log M_\varepsilon - \log u_\varepsilon}\ \left[\varepsilon^q - q\varepsilon^2 g_\varepsilon^{q-2} - (1-q) g_\varepsilon^q\right]. \tag{6.19}$$

If $q \geq 1$, we have $\varepsilon^q - q\varepsilon^2 g_\varepsilon^{q-2} \geq 0$ (as in Section 6.2.1), which, together with (6.18), implies $\tilde{R}_2^\varepsilon \geq 0$. We can simply drop this term and end up with

$$L_\varepsilon w_\varepsilon := \partial_t w_\varepsilon - A_\varepsilon w_\varepsilon - B_\varepsilon \cdot \nabla w_\varepsilon + C_2 w_\varepsilon^{(2+p_c)/2} - C_1 \varepsilon^{p_c-2\gamma} w_\varepsilon \leq 0$$

in Q_∞ by (6.8). We then argue as in Section 6.2.1 to check that, thanks to the choice of γ, the function

$$W_\varepsilon(t) = \left(\frac{2 + p_c C_1 \varepsilon^{p_c/2}}{p_c C_2}\right)^{2/p_c} t^{-2/p_c}$$

is a supersolution for the differential operator L_ε in $(0, \varepsilon^{(4\gamma-p_c)/2}) \times \mathbb{R}^N$. The comparison principle then ensures that

$$\left|\nabla u_\varepsilon^{-1/N}(t,x)\right| \leq C\left(1 + \varepsilon^{N/(N+1)}\right)^{1/p_c} (\log M_\varepsilon - \log u_\varepsilon(t,x))^{1/p_c} t^{-1/p_c} \tag{6.20}$$

for any $(t,x) \in (0, \varepsilon^{(4\gamma-p_c)/2}) \times \mathbb{R}^N$.

If $q \in (p_c/2, 1)$, we have to estimate $\tilde{R}_2^\varepsilon$ more precisely. Since the mapping $z \mapsto (2z-1)/z$ is increasing in $(0,\infty)$ and $\varepsilon^q - q\varepsilon^2 g_\varepsilon^{q-2} \geq (1-q)\varepsilon^q \geq 0$, it follows from (6.7), (6.18), and (6.19) that

$$\begin{aligned} \tilde{R}_2^\varepsilon w_\varepsilon &\geq -\frac{(1-q)(N+1)}{2Nu_\varepsilon}\ \frac{2\log M_\varepsilon - 2\log u_\varepsilon - 1}{\log M_\varepsilon - \log u_\varepsilon}\ g_\varepsilon^q\ w_\varepsilon \geq -C_3 u_\varepsilon^{-1} g_\varepsilon^q w_\varepsilon \\ &\geq -C_3 u_\varepsilon^{-1} (\varepsilon^2 + \varrho(u_\varepsilon)^2 w_\varepsilon)^{q/2} w_\varepsilon \geq -C_3 u_\varepsilon^{-1} (\varepsilon^q w_\varepsilon + \varrho(u_\varepsilon)^q w_\varepsilon^{(2+q)/2}) \\ &\geq -C_3 \varepsilon^{q-\gamma} w_\varepsilon - C_3 u_\varepsilon^{(q(N+1)-N)/N} (\log M_\varepsilon - \log u_\varepsilon)^{q(N+1)/2N} w_\varepsilon^{(2+q)/2}. \end{aligned}$$

We go on as in Section 2.3 by noticing that the function

$$z \mapsto z^{(q(N+1)-N)/N} (\log M_\varepsilon - \log z)^{q(N+1)/2N}$$

attains its maximum in the interval $(0,\|u_0\|_\infty+\varepsilon^\gamma)$ at $(\|u_0\|_\infty+\varepsilon^\gamma)e^{-(N\xi-1)/2}$, hence we can write:

$$\tilde{R}_2^\varepsilon \geq -C_4\varepsilon^{q-\gamma}w_\varepsilon - C_4\ (\|u_0\|_\infty+\varepsilon^\gamma)^{(q(N+1)-N)/N}\ w_\varepsilon^{(2+q)/2}.$$

It follows that

$$L_\varepsilon w_\varepsilon := \partial_t w_\varepsilon - A_\varepsilon w_\varepsilon - B_\varepsilon\cdot\nabla w_\varepsilon + C_2 w_\varepsilon^{(2+p_c)/2} - C_4(\|u_0\|_\infty+\varepsilon^\gamma)^{1/N\xi} w_\varepsilon^{1+q/2} - C_4\varepsilon^{q-\gamma}w_\varepsilon \leq 0$$

in Q_∞ since $\gamma < p_c-1 < p_c-q$. We notice that the function

$$\begin{aligned} W_\varepsilon(t) = {} & \left(\frac{4C_4}{C_2}\right)^{2/(p_c-q)}\ (\|u_0\|_\infty+\varepsilon^\gamma)^{2/(p_c-q)N\xi} + \left(\frac{4C_4}{C_2}\right)^{2/p_c}\ \varepsilon^{2(q-\gamma)/p_c} \\ & + \left(\frac{4}{p_c C_2}\right)^{2/p_c}\ t^{-2/p_c} \end{aligned}$$

is a supersolution in Q_∞. By the comparison principle, we obtain

$$\left|\nabla u_\varepsilon^{-1/N}(t,x)\right| \leq C\ \left[(\|u_0\|_\infty+\varepsilon^\gamma)^{1/(p_c-q)N\xi} + \varepsilon^{(q-\gamma)/p_c} + t^{-1/p_c}\right]\ \left(\log\left(\frac{M_\varepsilon}{u_\varepsilon(t,x)}\right)\right)^{1/p_c}$$

for any $(t,x)\in Q_\infty$.

For $q=p_c/2$, following the idea in Section 2.3, we choose

$$\varrho(z) = z^{(N+1)/N}(\log M_\varepsilon - \log z)^{(N+1)/N}, \quad M_\varepsilon = e(\|u_0\|_\infty+\varepsilon^\gamma).$$

Proceeding as in the previous cases, we infer from (6.7), (6.13), and (6.18) that

$$\begin{aligned} R_{11}^\varepsilon w_\varepsilon &\geq -\frac{2(N+1)}{N^2}\ \varepsilon^{-2\gamma}\ g_\varepsilon^{p_c-2}, \\ R_1^\varepsilon &\geq \frac{N+1}{N}\ u_\varepsilon^{2/N}\ (\log M_\varepsilon - \log u_\varepsilon)^{(N+2)/N}\ g_\varepsilon^{p_c-2}, \end{aligned}$$

so that

$$\tilde{R}_1^\varepsilon\ w_\varepsilon^2 \geq \frac{N-1}{N}\ (\log M_\varepsilon - \log u_\varepsilon)\ w_\varepsilon^{(2+p_c)/2} - C_1\ \varepsilon^{p_c-2\gamma}\ w_\varepsilon,$$

while

$$\tilde{R}_2^\varepsilon\ w_\varepsilon \geq -\frac{\varepsilon^{q-\gamma}}{N}\ w_\varepsilon - \frac{1}{N}\ (\log M_\varepsilon - \log u_\varepsilon)\ w_\varepsilon^{(2+q)/2}.$$

Using a comparison argument as before we end up with the following estimate

$$\left|\nabla u_\varepsilon^{-1/N}(t,x)\right| \leq C\ (\log M_\varepsilon - \log u_\varepsilon(t,x))^{(N+1)/N}\ \left(1+\varepsilon^{(q-\gamma)/p_c}+t^{-1/p_c}\right)$$

for any $(t,x)\in Q_\infty$.

Finally, if $q\in(0,p_c/2)$, we proceed as in Section 6.2.2 to show that (6.17) holds true.

6.2.4 $p < p_c$ and $q > 1-k$.

We slightly modify the function ϱ from the formal proof in Section 2.4 and define the function ϱ_ε by

$$\left(\frac{(2-p-2k)K_\varepsilon^p}{2}\right)^{1/2} \int_0^{\varrho_\varepsilon(r)/K_\varepsilon} \frac{dz}{z^k\,(1-z^{2-p-2k})^{1/2}} = r \tag{6.21}$$

for $r \in [0, \|u_0\|_\infty + \varepsilon^\gamma]$, where

$$\left(\frac{(2-p-2k)K_\varepsilon^p}{2}\right)^{1/2} \int_0^1 \frac{dz}{z^k\,(1-z^{2-p-2k})^{1/2}} = \|u_0\|_\infty + \varepsilon^\gamma\,. \tag{6.22}$$

Observe that $K_\varepsilon = \kappa\,(\|u_0\|_\infty + \varepsilon^\gamma)^{2/p}$ and $K_\varepsilon \to K_0$ as $\varepsilon \to 0$, the constants κ and K_0 being defined in (2.20). It readily follows from (6.21) that ϱ_ε solves (2.19) with K_ε instead of K_0 and thus (2.18) and

$$\varrho_\varepsilon(r) \le C\ K_\varepsilon^{(2-p-2k)/2(1-k)}\ r^{1/(1-k)}\,, \quad r \in [0, \|u_0\|_\infty + \varepsilon^\gamma]\,. \tag{6.23}$$

Now, omitting as before the last term in R_{11}^ε since it is non-negative, we deduce from (2.18) and (2.19) that

$$\begin{aligned} R_{11}^\varepsilon &\ge \left[(2-p)\ \left(k\varrho_\varepsilon'(u_\varepsilon)^2 - \varrho_\varepsilon''(u_\varepsilon)\varrho_\varepsilon(u_\varepsilon)\right) - k\ \varrho_\varepsilon'(u_\varepsilon)^2\right]\ g_\varepsilon^{p-4} \\ &\ge \left[(2-p)\ \varrho_\varepsilon(u_\varepsilon)^{2-p} - C\ K_\varepsilon^{2-p-2k}\ \varrho_\varepsilon(u_\varepsilon)^{2k}\right]\ g_\varepsilon^{p-4} \\ &\ge -C\ K_\varepsilon^{2-p-2k}\ \varrho_\varepsilon(u_\varepsilon)^{2k}\ g_\varepsilon^{p-4}. \end{aligned}$$

We then infer from (6.12), (6.13), (6.23), and the positivity of $2-p-2k>0$ that

$$R_{11}^\varepsilon\ w_\varepsilon \ge -C\ K_\varepsilon^{2-p-2k}\ \varrho_\varepsilon(u_\varepsilon)^{2k-2}\ g_\varepsilon^{p-4}\ \left(g_\varepsilon^2 - \varepsilon^2\right) \ge -C\ K_\varepsilon^{2-p-2k}\ \varrho_\varepsilon(u_\varepsilon)^{2k-2}\ g_\varepsilon^{p-2}\,. \tag{6.24}$$

Since $k<1$ and ϱ_ε is increasing, we deduce from (6.7) that

$$\varrho_\varepsilon(u_\varepsilon)^{2k-2} \le \varrho_\varepsilon\left(\varepsilon^\gamma\right)^{2k-2}\,. \tag{6.25}$$

Now, on the one hand, as $2-p-2k>0$, we deduce from (6.21) that

$$\begin{aligned} \varepsilon^\gamma &= \left(\frac{(2-p-2k)K_\varepsilon^p}{2}\right)^{1/2} \int_0^{\varrho_\varepsilon(\varepsilon^\gamma)/K_\varepsilon} \frac{dz}{z^k\,(1-z^{2-p-2k})^{1/2}} \\ &\le \left(\frac{2-p-2k}{2}\right)^{1/2}\ K_\varepsilon^{1-k} \int_0^{\varrho_\varepsilon(\varepsilon^\gamma)/K_\varepsilon} \frac{dz}{z^k\left(K_\varepsilon^{2-p-2k} - \varrho_\varepsilon(\varepsilon^\gamma)^{2-p-2k}\right)^{1/2}} \\ &\le \left(\frac{2-p-2k}{2(1-k)^2}\right)^{1/2}\ \frac{\varrho_\varepsilon(\varepsilon^\gamma)^{1-k}}{\left(K_\varepsilon^{2-p-2k} - \varrho_\varepsilon(\varepsilon^\gamma)^{2-p-2k}\right)^{1/2}} \end{aligned}$$

On the other hand, using again the positivity of $2-p-2k$ and $1-k$ and (6.23), we find that

$$\varrho_\varepsilon(\varepsilon^\gamma) \le C\ K_\varepsilon^{(2-p-2k)/2(1-k)}\ \varepsilon^{\gamma/(1-k)} \le \frac{1}{2^{1/(2-p-2k)}}\ K_\varepsilon\,, \tag{6.26}$$

provided $\varepsilon \le \varepsilon_0(\|u_0\|_\infty)$ is chosen suitably small. Combining (6.25) and (6.26) yields

$$\varepsilon^\gamma \le \left(\frac{2-p-2k}{(1-k)^2} \right)^{1/2} \varrho_\varepsilon(\varepsilon^\gamma)^{1-k}\ K_\varepsilon^{-(2-p-2k)/2}\,.$$

Consequently,

$$\varrho_\varepsilon(\varepsilon^\gamma) \ge C\ \varepsilon^{\gamma/(1-k)}\ K_\varepsilon^{(2-p-2k)/2(1-k)} \tag{6.27}$$

which, together with (6.12), (6.24), and (6.25) gives

$$R_{11}^\varepsilon\ w_\varepsilon \ge -C\ K_\varepsilon^{2-p-2k}\ K_\varepsilon^{-(2-p-2k)}\ \varepsilon^{-2\gamma}\ g_\varepsilon^{p-2} \ge -C\ \varepsilon^{p-2-2\gamma}\,.$$

Turning to R_1^ε, it follows from (2.18), (6.7), (6.12), (6.13), the monotonicity of ϱ_ε, and (6.27) that

$$\begin{aligned}
R_1^\varepsilon\ w_\varepsilon^2 &= \varrho_\varepsilon(u_\varepsilon)^{2-p}\ g_\varepsilon^{p-2}\ w_\varepsilon^2 = \varrho_\varepsilon(u_\varepsilon)^{-p}\ g_\varepsilon^{p-2}\ \left(g_\varepsilon^2 - \varepsilon^2\right)\ w_\varepsilon \\
&\ge \varrho_\varepsilon(u_\varepsilon)^{-p}\ g_\varepsilon^p\ w_\varepsilon - \varepsilon^p\ \varrho_\varepsilon(u_\varepsilon)^{-p}\ \ w_\varepsilon \ge w_\varepsilon^{(2+p)/2} - \varepsilon^p\ \varrho_\varepsilon(\varepsilon^\gamma)^{-p}\ \ w_\varepsilon \\
&\ge w_\varepsilon^{(2+p)/2} - C\ \varepsilon^{p(1-k-\gamma)/(1-k)}\ K_\varepsilon^{-p(2-p-2k)/(2-2k)}\ \ w_\varepsilon\,.
\end{aligned}$$

Gathering the above lower bounds on R_1^ε and R_{11}^ε, we are lead to

$$\tilde{R}_1^\varepsilon\ w_\varepsilon^2 \ge 2(p-1)\ w_\varepsilon^{(2+p)/2} - C\left[\varepsilon^{p(1-k-\gamma)/(1-k)}\ K_\varepsilon^{-p(2-p-2k)/(2-2k)} + \varepsilon^{p-2\gamma}\right]\ w_\varepsilon.$$

For $q \ge 1$, the influence of $\tilde{R}_2^\varepsilon$ is a positive term thanks to the monotonicity of ϱ_ε (as in the previous cases) and can be omitted. We obtain that

$$L_\varepsilon w_\varepsilon := \partial_t w_\varepsilon - A_\varepsilon w_\varepsilon - B_\varepsilon\cdot\nabla w_\varepsilon + 2(p-1)\ w_\varepsilon^{(2+p)/2} - C_1\ \mu_\varepsilon^2\ w_\varepsilon \le 0 \quad \text{in} \quad Q_\infty\,,$$

with

$$\mu_\varepsilon := \varepsilon^{p(1-k-\gamma)/(1-k)}\ K_\varepsilon^{-p(2-p-2k)/(2-2k)} + \varepsilon^{p-2\gamma} \underset{\varepsilon\to 0}{\longrightarrow} 0 \tag{6.28}$$

thanks to the choice of γ. By noticing that

$$W_\varepsilon(t) = \left(\frac{1+pC_1\mu_\varepsilon}{2p(p-1)} \right)^{2/p} t^{-2/p}\,, \qquad t>0\,,$$

is a supersolution for the differential operator L_ε in $(0,\mu_\varepsilon^{-1})\times\mathbb{R}^N$. The comparison principle then implies that

$$|\nabla u_\varepsilon(t,x)| \le C\ \varrho_\varepsilon(u_\varepsilon(t,x))\ (1+\mu_\varepsilon)^{1/p}\ t^{-1/p}, \quad (t,x)\in(0,\mu_\varepsilon^{-1})\times\mathbb{R}^N)\,,$$

whence

$$|\nabla u_\varepsilon(t,x)| \le C\ \left(\|u_0\|_\infty + \varepsilon^\gamma\right)^{(2-p-2k)/p(1-k)}\ u_\varepsilon(t,x)^{1/(1-k)}\ (1+\mu_\varepsilon)^{1/p}\ t^{-1/p} \tag{6.29}$$

for any $(t,x)\in(0,\mu_\varepsilon^{-1})\times\mathbb{R}^N$. This is the approximation giving, in the limit, the estimates in Section 2.4.

For $q \in [1-k,1)$, we necessarily have $p > p_{sc} = 2(N+1)/(N+3)$ and, recalling that $k<1$, it follows from (2.19), (6.7), (6.12), (6.27), and the monotonicity of ϱ_ε that

$$
\begin{aligned}
\tilde{R}_2^\varepsilon\ w_\varepsilon &\ge -(1-q)\ \frac{\varrho_\varepsilon'(u_\varepsilon)}{\varrho_\varepsilon(u_\varepsilon)}\ g_\varepsilon^q\ w_\varepsilon \ge -C\ K_\varepsilon^{(2-p-2k)/2}\ \varrho_\varepsilon(u_\varepsilon)^{k-1}\ g_\varepsilon^q\ w_\varepsilon \\
&\ge -C\ K_\varepsilon^{(2-p-2k)/2}\ \varrho_\varepsilon(u_\varepsilon)^{k-1}\ \left(\varepsilon^q + \varrho_\varepsilon(u_\varepsilon)^q\ w_\varepsilon^{q/2}\right)\ w_\varepsilon \\
&\ge -C\ K_\varepsilon^{(2-p-2k)/2}\ \left[\varrho_\varepsilon(\varepsilon^\gamma)^{k-1}\ \varepsilon^q\ w_\varepsilon + \varrho_\varepsilon(u_\varepsilon)^{q+k-1}\ w_\varepsilon^{(2+q)/2}\right] \\
&\ge -C\ \varepsilon^{q-\gamma}\ w_\varepsilon - C\ K_\varepsilon^{(2-p-2k)/2}\ \varrho_\varepsilon(\|u_0\|_\infty + \varepsilon^\gamma)^{q+k-1}\ w_\varepsilon^{(2+q)/2} \\
&\ge -C_2\ \left[\varepsilon^{q-\gamma}\ w_\varepsilon + K_\varepsilon^{(2q-p)/2}\ w_\varepsilon^{(2+q)/2}\right]\ .
\end{aligned}
$$

Combining this lower bound with that for $\tilde{R}_1^\varepsilon\ w_\varepsilon^2$ established above, we realize that

$$
\begin{aligned}
L_\varepsilon w_\varepsilon := \partial_t w_\varepsilon - A_\varepsilon w_\varepsilon - B_\varepsilon \cdot \nabla w_\varepsilon + 2(p-1)\ w_\varepsilon^{(2+p)/2} & \\
- C_2\ K_\varepsilon^{(2q-p)/2}\ w_\varepsilon^{(2+q)/2} - \left(C_1\ \mu_\varepsilon^2 + C_2\ \varepsilon^{q-\gamma}\right)\ w_\varepsilon &\le 0
\end{aligned}
$$

in Q_∞ with μ_ε defined by (6.28). We next observe that the function

$$
W_\varepsilon(t) = \left(C_1\ \mu_\varepsilon^2 + C_2\ \varepsilon^{q-\gamma}\right)^{2/p} + \left(\frac{C_2 K_\varepsilon^{(2q-p)/2}}{p-1}\right)^{2/(p-q)} + \left(\frac{2}{p(p-1)}\right)^{2/p}\ t^{-2/p}
$$

is a supersolution for the differential operator L_ε in Q_∞ and deduce from the comparison principle and (6.23) that

$$
\begin{aligned}
& |\nabla u_\varepsilon(t,x)|\ u_\varepsilon(t,x)^{-1/(1-k)}\ \left(\|u_0\|_\infty + \varepsilon^\gamma\right)^{-(2-p-2k)/p(1-k)} \\
& \le C\ \left(\mu_\varepsilon^{2/p} + \varepsilon^{(q-\gamma)/p} + \left(\|u_0\|_\infty + \varepsilon^\gamma\right)^{(2q-p)/p(p-q)} + t^{-1/p}\right)
\end{aligned}
\tag{6.30}
$$

for any $(t,x) \in Q_\infty$.

6.3 A gradient estimate related to the Hamilton-Jacobi term

We prove, using the same approximation as before, the gradient estimates (1.28) and (1.29) formally established in Section 2.6. As already mentioned, we assume for simplicity $p > p_{sc} = 2(N+1)/(N+3)$ and divide the proof into two cases.

6.3.1 $q \in (0,1)$.

We set $\varrho(z) = -2(M_\varepsilon - z)^{1/2}$ for $z \in [0, M_\varepsilon]$, where $M_\varepsilon := \|u_0\|_\infty + 2\varepsilon^\gamma$. On the one hand, we have

$$
\begin{aligned}
R_1^\varepsilon &= (k+1)\ g_\varepsilon^{p-2}\ (M_\varepsilon - u_\varepsilon)^{-1} \ge 0\,, \\
R_{11}^\varepsilon &\ge \frac{(N+3)(2-p)^2}{4}\ g_\varepsilon^{p-4}\ (M_\varepsilon - u_\varepsilon)^{-1} \ge 0\,.
\end{aligned}
$$

On the other hand, by (6.7), we have

$$\begin{aligned}
R_2^\varepsilon w_\varepsilon &= \frac{1}{2}(M_\varepsilon - u_\varepsilon)^{-1} \left[(1-q)g_\varepsilon^q + q\varepsilon^2 g_\varepsilon^{q-2} - \varepsilon^q\right] w_\varepsilon \\
&\geq \frac{1-q}{2}(M_\varepsilon - u_\varepsilon)^{-1} \left(\varepsilon^2 + \varrho(u_\varepsilon)^2 w_\varepsilon\right)^{q/2} w_\varepsilon - \varepsilon^q \ (M_\varepsilon - u_\varepsilon)^{-1} \ w_\varepsilon \\
&\geq \frac{1-q}{2} \ (M_\varepsilon - u_\varepsilon)^{-1} \ |\varrho(u_\varepsilon)|^q \ w_\varepsilon^{(2+q)/2} - \varepsilon^{q-\gamma} \ w_\varepsilon \\
&\geq 2^{q-1}(1-q) \ (M_\varepsilon - u_\varepsilon)^{(q-2)/2} \ w_\varepsilon^{(2+q)/2} - \varepsilon^{q-\gamma} \ w_\varepsilon \\
&\geq 2^{q-1}(1-q) \ (\varepsilon^\gamma + \|u_0\|_\infty)^{(q-2)/2} \ w_\varepsilon^{(2+q)/2} - \varepsilon^{q-\gamma} \ w_\varepsilon,
\end{aligned}$$

where we have used the bounds $\varepsilon^\gamma \leq M_\varepsilon - u_\varepsilon \leq \varepsilon^\gamma + \|u_0\|_\infty$. Therefore

$$L_\varepsilon w_\varepsilon := \partial_t w_\varepsilon - A_\varepsilon w_\varepsilon - B_\varepsilon \cdot \nabla w_\varepsilon + C_1 \ (\varepsilon^\gamma + \|u_0\|_\infty)^{(q-2)/2} \ w_\varepsilon^{(2+q)/2} - \varepsilon^{q-\gamma} \ w_\varepsilon \leq 0$$

in Q_∞. Now, the following function

$$W_\varepsilon(t) := (\varepsilon^\gamma + \|u_0\|_\infty)^{(2-q)/q} \left(\frac{2 + q\varepsilon^{q/2}}{qC_1}\right)^{2/q} t^{-2/q}, \quad t > 0,$$

is a supersolution for the differential operator L_ε in $(0, \varepsilon^{(2\gamma-q)/2}) \times \mathbb{R}^N$. We then deduce from the comparison principle that

$$\begin{aligned}
|\nabla u_\varepsilon(t,x)| &\leq C \ (M_\varepsilon - u_\varepsilon(t,x))^{1/2} \ (\varepsilon^\gamma + \|u_0\|_\infty)^{(2-q)/2q} \ \left(1 + \varepsilon^{q/2}\right)^{1/q} \ t^{-1/q} \\
&\leq C \ (\varepsilon^\gamma + \|u_0\|_\infty)^{1/q} \ \left(1 + \varepsilon^{q/2}\right)^{1/q} \ t^{-1/q}
\end{aligned}$$

for any $(t,x) \in (0, \varepsilon^{(2\gamma-q)/2}) \times \mathbb{R}^N$.

6.3.2 $q > 1$.

We set $\varrho(z) = z^{1/q}$ for $z \geq 0$. Owing to (6.12), we have

$$\begin{aligned}
\tilde{R}_1^\varepsilon &\geq \frac{u_\varepsilon^{(2-q)/2}}{q^2} \ \left[(p-1)(k+q-1) \ g_\varepsilon^{p-2} + \varepsilon^2 \ \left((2-p)(q-1) - k(p-1)\right) \ g_\varepsilon^{p-4}\right] \\
&\geq \varepsilon^2 \ \frac{u_\varepsilon^{(2-q)/2}}{q^2} \ \left[(p-1)(k+q-1) + (2-p)(q-1) - k(p-1)\right] \ g_\varepsilon^{p-4} \\
&\geq \varepsilon^2 \ \frac{(q-1) \ u_\varepsilon^{(2-q)/2}}{q^2} \ g_\varepsilon^{p-4} \geq 0.
\end{aligned}$$

Since we are interested only in the effect of the Hamilton-Jacobi part, we omit this term. Next, arguing as in [3] and using (6.7), we obtain

$$\begin{aligned}
\tilde{R}_2^\varepsilon \ w_\varepsilon &= \frac{1}{qu_\varepsilon} \ \left[(q-1)g_\varepsilon^q + \varepsilon^q - q\varepsilon^2 g_\varepsilon^{q-2}\right] w_\varepsilon \geq \frac{\min\{1, q-1\}}{qu_\varepsilon}(g_\varepsilon^q - \varepsilon^q) \ w_\varepsilon \\
&\geq C_1 \ u_\varepsilon^{-1} \ \varrho(u_\varepsilon)^q \ w_\varepsilon^{(2+q)/2} - \varepsilon^q \ u_\varepsilon^{-1} \ w_\varepsilon \geq C_1 \ w_\varepsilon^{(2+q)/2} - \varepsilon^{q-\gamma} \ w_\varepsilon.
\end{aligned}$$

We obtain that

$$L_\varepsilon w_\varepsilon := \partial_t w_\varepsilon - A_\varepsilon w_\varepsilon - B_\varepsilon \cdot \nabla w_\varepsilon + C_1 \ w_\varepsilon^{(2+q)/2} - \varepsilon^{q-\gamma} \ w_\varepsilon \leq 0$$

in Q_∞. Following the same computations as in Section 6.2, we notice that the function

$$W_\varepsilon(t) := \left(\frac{2+q\varepsilon^{q/2}}{qC_1} \right)^{2/q} t^{-2/q}\,, \quad t>0\,,$$

is a supersolution for the differential operator L_ε in $(0,\varepsilon^{(2\gamma-q)/2})\times\mathbb{R}^N$. We then infer from the comparison principle that

$$\left|\nabla u_\varepsilon^{(q-1)/q}(t,x)\right| \le \frac{q-1}{q} \left(\frac{2+q\varepsilon^{q/2}}{qC_1} \right)^{1/q} t^{-1/q} \tag{6.31}$$

for any $(t,x)\in(0,\varepsilon^{(2\gamma-q)/2})\times\mathbb{R}^N$.

6.4 Existence

We have to pass to the limit as $\varepsilon\to 0$, and to this aim we follow the lines of [2, Section 3]. The uniform gradient bound (6.11) ensures that the family u_ε is equicontinuous with respect to the space variable and we next argue as in [14, Lemma 5] to establish the time equicontinuity. As a consequence, we are in a position to apply the Arzelà-Ascoli theorem and conclude that there exists a limit

$$u(t,x) := \lim_{\varepsilon\to 0} u_\varepsilon(t,x),$$

with uniform convergence in compact subsets of $[0,\infty)\times\mathbb{R}^N$. By the stability result for viscosity solutions [24, Theorem 6.1], we conclude that u is a viscosity solution for the equation (1.1) with initial condition u_0, satisfying moreover that

$$0\le u(t,x)\le \|u_0\|_\infty.$$

Finally, the dependence on ε in the right-hand side of the approximate gradient estimates (6.15)-(6.30) (depending on the range of the exponents p and q) and in the time interval validity of these estimates allow us to pass to the limit in an uniform way, while in the left-hand side we can pass to the limit in the gradient terms in the weak sense. We thus end the proof of the gradient estimates in Theorems 1.3, 1.5 and 1.7. In addition, using [10, Theorem 4.1], it can be shown (as in [2]) that

$$\nabla u_\varepsilon \to \nabla u \quad \text{a. e. in } Q_\infty\,,$$

so that u is also a weak solution to (1.1) and satisfies (6.2) and also (6.3). Finally, the uniqueness assertion follows from [24, Theorem 3.1].

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References

[1] D. Andreucci, A. F. Tedeev, and M. Ughi, *The Cauchy problem for degenerate parabolic equations with source and damping*, Ukrainian Math. Bull., **1** (2004), 1–23.

[2] J.-Ph. Bartier and Ph. Laurençot, *Gradient estimates for a degenerate parabolic equation with gradient absorption and applications*, J. Funct. Anal., **254** (2008), 851–878.

[3] S. Benachour and Ph. Laurençot, *Global solutions to viscous Hamilton-Jacobi equations with irregular initial data*, Comm. Partial Differential Equations, **24** (1999), no. 11-12, 1999–2021.

[4] S. Benachour, G. Karch, and Ph. Laurençot, *Asymptotic profiles of solutions to viscous Hamilton-Jacobi equations*, J. Math. Pures Appl., **83** (2004), 1275–1308.

[5] S. Benachour, Ph. Laurençot, and D. Schmitt, *Extinction and decay estimates for viscous Hamilton-Jacobi equations in* $\mathbb{R}^N$, Proc. Amer. Math. Soc., **130** (2001), no. 4, 1103–1111.

[6] S. Benachour, Ph. Laurençot, D. Schmitt, and Ph. Souplet, *Extinction and non-extinction for viscous Hamilton-Jacobi equations in* $\mathbb{R}^N$, Asympt. Anal., **31** (2002), 229–246.

[7] S. Benachour, B. Roynette, and P. Valois, *Asymptotic estimates of solutions of* $u_t - \frac{1}{2}\Delta u = -|\nabla u|$ *in* $\mathbb{R}_+ \times \mathbb{R}^d$, $d \geq 2$, J. Funct. Anal., **144** (1997), 301–324.

[8] Ph. Bénilan, *Evolution equations and accretive operators*, Lecture Notes taken by S. Lenhardt, Univ. of Kentucky, 1981.

[9] P. Biler, M. Guedda, and G. Karch, *Asymptotic properties of solutions of the viscous Hamilton-Jacobi equation*, J. Evolution Equations, **4** (2004), 75–97.

[10] L. Boccardo and F. Murat, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Anal. **19** (1992), 581–597.

[11] M. Bonforte, R. Iagar, and J. L. Vázquez, *Local smoothing effects, positivity and Harnack inequalities for the fast p-Laplacian equation*, Adv. Math., **224** (2010), no. 5, 2151–2215.

[12] Th. Gallay and Ph. Laurençot, *Asymptotic behavior for a viscous Hamilton-Jacobi equation with critical exponent*, Indiana Univ. Math. J., **56** (2007), 459–479.

[13] B. H. Gilding, *The Cauchy problem for* $u_t = \Delta u + |\nabla u|^q$, *large-time behaviour*, J. Math. Pures Appl., **84** (2005), 753–785.

[14] B. H. Gilding, M. Guedda, and R. Kersner, *The Cauchy problem for* $u_t = \Delta u + |\nabla u|^q$, J. Math. Anal. Appl., **284** (2003), no. 2, 733–755.

[15] R. S. Hamilton, *A matrix Harnack estimate for the heat equation*, Comm. Anal. Geom. **1** (1993), no. 1, 113–126.

[16] M. A. Herrero and J. L. Vázquez, *Asymptotic behaviour of the solutions of a strongly nonlinear parabolic problem*, Ann. Fac. Sci. Toulouse Math., **3** (1981), 113–127.

[17] R. Iagar, Ph. Laurençot, and J. L. Vázquez, *Asymptotic behavior of a nonlinear parabolic equation with gradient absorption and critical exponent*, Interfaces Free Bound., to appear.

[18] R. Iagar, A. Sánchez, and J. L. Vázquez, *Radial equivalence for the two basic nonlinear degenerate diffusion equations*, J. Math. Pures Appl., **89** (2008), no. 1, 1–24.

[19] H. Ishii and P. E. Souganidis, *Generalized motion of noncompact hypersurfaces with velocity having arbitrary growth on the curvature tensor*, Tohoku Math. J., **47** (1995), 227–250.

[20] P. Juutinen, P. Lindqvist, and J. J. Manfredi, *On the equivalence of the viscosity solutions and weak solutions for a quasi-linear equation*, SIAM J. Math. Anal., **33** (2001), no. 3, 699–717.

[21] Ph. Laurençot, *Large time behavior for diffusive Hamilton-Jacobi equations*, in "Topics in Mathematical Modeling", Jindrich Necas Center for Mathematical Modeling Lecture Notes, vol. 4, Praha, 2008.

[22] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Uraltseva, *Linear and Quasi-linear Equations of Parabolic Type*, Amer. Math. Society Trans. Math. Monographs, no. 23, 1968.

[23] Ph. Laurençot and J. L. Vázquez, *Localized non-diffusive asymptotic patterns for nonlinear parabolic equations with gradient absorption*, J. Dynamics Differential Equations, **19** (2007), 985–1005.

[24] M. Ohnuma and K. Sato, *Singular degenerate parabolic equations with applications to the p-Laplace diffusion equation*, Comm. Partial Differential Equations, **22** (1997), no. 3-4, 381–411.

[25] J. L. Vázquez, *Smoothing and Decay Estimates for Nonlinear Diffusion Equations. Equations of Porous Medium Type*, Oxford University Press, Oxford, 2006.